EXAM 1 REVIEW SOLUTIONS

Problem 1. Prove De Morgan’s laws:

\[(A \cap B)^c = A^c \cup B^c\]
\[(A \cup B)^c = A^c \cap B^c.\]

Proof. The main idea of these proofs is that negation changes “and” into “or,” and vice versa. Therefore, we only prove the first law:

\((\subseteq)\) Suppose \(x \in (A \cap B)^c\). This means \(x \not\in A \cap B\). Notice that the negation of “\(x \in A\) and \(x \in B\)” is equivalent to “\(x \not\in A\) or \(x \not\in B\).” This implies that \(x \in A^c\) or \(x \in B^c\). In other words, \(x \in A^c \cup B^c\).

\((\supseteq)\) Suppose \(x \in A^c \cup B^c\). This means \(x \not\in A\) or \(x \not\in B\). This is logically equivalent to the negation of “\(x \in A\) and \(x \in B\)” in other words, it is equivalent to the negation of \(x \in A \cap B\). We may conclude that \(x \not\in A \cap B\), i.e., \(x \in (A \cap B)^c\).

Problem 2. Suppose \(x > 0\). Prove or give a counterexample: if \(x\) is irrational, then \(\sqrt{x}\) is irrational.

Proof. Let us prove the contrapositive: if \(\sqrt{x}\) is rational, then \(x\) is rational. (Here, \(x > 0\) is given.)

Suppose \(\sqrt{x}\) is rational, so \(\sqrt{x} = a/b\) for some \(a, b \in \mathbb{Z}\), \(b \neq 0\). We compute

\[x = (\sqrt{x})^2 = \frac{a^2}{b^2}.\]

Since \(a^2, b^2 \in \mathbb{Z}\) with \(b^2 \neq 0\), \(x\) has the form of a rational number. We conclude that \(x\) is rational whenever \(\sqrt{x}\) is rational.

Problem 3 (Lay 1.2.13, 15). Write the following defining conditions using logical symbolism and then negate:

13. A function \(f\) is even if for every \(x\), \(f(-x) = f(x)\).
15. A function \(f\) is increasing if for every \(x\) and \(y\), if \(x \leq y\), then \(f(x) \leq f(y)\).

Solution. 13. \(\forall x, f(-x) = f(x)\).

Negation: \(\exists x\) s.t. \(f(-x) \neq f(x)\).

15. \(\forall x \forall y, x \leq y \implies f(x) \leq f(y)\).

Negation: \(\exists x \exists y\) s.t. \(x \leq y \land f(x) > f(y)\).

Problem (Lay 2.1.24 d,e). Prove the distributive laws for sets:

\[A \cup (B \cap C) = (A \cup B) \cap (A \cup C)\]
\[A \cap (B \cup C) = (A \cap B) \cup (A \cap C).\]
Proof. The main idea is that the logical “and” and “or” distribute across each other (we proved that using truth tables a while back). We just prove the first law. The second law is similar.

\((\subseteq)\) Suppose \(x \in A \cup (B \cap C)\). This means \(x \in A\) or \((x \in B \text{ and } x \in C)\). This is logically equivalent to \(((x \in A \text{ or } x \in B) \text{ and } (x \in A \text{ or } x \in C))\). In other words, \(x \in A \cup B\) and \(x \in A \cup C\), which may be rewritten as \(x \in (A \cup B) \cap (A \cup C)\).

\((\supseteq)\) Suppose \(x \in (A \cup B) \cap (A \cup C)\). This means \(((x \in A \text{ or } x \in B) \text{ and } (x \in A \text{ or } x \in C))\), which is logically equivalent to \((x \in A \text{ or } (x \in B \text{ and } x \in C))\). This last statement means precisely that \(x \in A \cup (B \cap C)\).

Problem (Lay 2.2.11). I am not retyping all of these. See p. 60 of the 5th edition.

Solution. (a) Reflexive and transitive, but not symmetric
(b) Reflexive and transitive, but not symmetric
(c) None. (It’s not reflexive unless everybody “likes themselves,” so that’s kind of subjective.)
(d) Reflexive and transitive, but not symmetric
(e) Reflexive, symmetric, and transitive. It is an equivalence relation! The equivalence classes are really wonky.
(f) Symmetric, but not reflexive or transitive
(g) This is the empty relation on \(\mathbb{R}\). It is not reflexive, symmetric, or transitive.
(h) Reflexive and symmetric, but not transitive

Problem (Lay 2.2.13). Let \(S\) be the Cartesian coordinate plane \(\mathbb{R} \times \mathbb{R}\) and define a relation \(R\) on \(S\) by \((a, b)R(c, d)\) iff \(a = c\). Verify that \(R\) is an equivalence relation and describe a typical equivalence class \(E_{(a, b)}\).

Solution. First, we prove that \(R\) is reflexive. Let \((a, b) \in S\). Since \(a = a\), we have \((a, b)R(a, b)\).

Next, we prove that \(R\) is symmetric. Let \((a, b), (c, d) \in S\). Suppose \((a, b)R(c, d)\), i.e., \(a = c\). This implies \(c = a\), so \((c, d)R(a, b)\).

Lastly, we prove that \(R\) is transitive. Let \((a, b), (c, d), (e, f) \in S\) and suppose \((a, b)R(c, d)\) and \((c, d)R(e, f)\). This means \(a = c\) and \(c = e\), which implies \(a = e\). Therefore, \((a, b)R(e, f)\).

Equivalence classes are the sets of points which have the same “\(x\) coordinate”:

\[E_{(a, b)} = \{(c, d) \in \mathbb{R}^2 \mid a = c\}\]

In other words, the equivalence classes are vertical lines in \(\mathbb{R}^2\). 

\[1\text{Parentheses needed here.}\]