Math 8501 — HW III
due Wednesday, December 14

1. (Variational Equations in $\mathbf{R}^1$) Consider the scalar ODE $\dot{x} = f(x) = x(1-x)$, the well-known “logistic” equation.
   a. Show that the flow is given by
      $\phi(t, x) = \frac{x e^t}{1 - x + x e^t}$.
   b. Write down the variational equation $\dot{u} = Df(\phi(t, x))u$ for this system and find the solution $u(t, x)$ with $u(0, x) = 1$. Note: In this linear ODE, all the matrices are $1 \times 1$, so it can be solved by elementary methods.
   c. Verify that $u(t, x) = D\phi(t, x)$ where $D$ denotes the $x$-derivative.

2. (Variational Equations in Outer Space (V.V. Beletski and V.I. Arnold)) During a spacewalk an astronaut in a circular orbit around the earth throws the lens cap of his camera directly at the earth below him. This problem uses variational equations to approximate the subsequent motion of the lens cap.

   Consider the Kepler problem in the plane, given by the second-order ODE $\ddot{q} = -\frac{q}{r^3}$ where $q \in \mathbf{R}^2$ is the position of the orbiting particle. Using polar coordinates in the plane one gets the differential equations:
   \[
   \ddot{r} - r \dot{\theta}^2 = -\frac{1}{r^2} \quad r \ddot{\theta} + 2 \dot{r} \dot{\theta} = 0.
   \]
   a. Show that there are circular orbits where $r(t) = r_0$ is constant and $\theta(t) = ct$. Find the relation between $r_0$ and $c$ for such orbits. We will assume our astronaut is in the circular orbit with $r_0 = c = 1$.
   b. Write the ODE as a first order system with variables $(r, v, \theta, w) = (r, \dot{r}, \dot{\theta}, \dot{w})$. Find the variational ODE of this system along the circular orbit from part a. This will be a linear ODE approximating the behavior of the difference vector $(\delta r, \delta v, \delta \theta, \delta w)$ between the astronaut and the lens cap.
   c. Solve the variational equations with the appropriate initial conditions to find the (rather surprising) behavior of the lens cap. In particular, find $\delta r(t)$ and $\delta \theta(t)$ and try to visualize the motion of the lens cap.
   Hint: Show that the variational ODE leads to the simple scalar equation $\ddot{v} + v = 0$.

3. (Variational Equations with respect to a parameter). The following ODE is a special case of the Van der Pol equation:
   \[
   \ddot{q} + \epsilon(q^2 - 1)\dot{q} + q = 0 \quad q \in \mathbf{R}^1.
   \]
   Of course there is an equivalent first-order system $\dot{x} = f(x, \epsilon)$ in $\mathbf{R}^2$ with variables $x = (q, v) = (q, \dot{q})$.
   a. When $\epsilon = 0$ we have a harmonic oscillator equation which is easily solve. In particular there are circular orbits with initial conditions $(q(0), v(0)) = (k, 0)$ for any $k > 0$. Find formulas for these solutions.
   b. Consider the solution of the VdP equation with the same initial condition $(q(0), v(0)) = (k, 0)$, but for $\epsilon > 0$. Solve the variational ODE with respect to parameters along the circular solution from part a to find the approximate behavior of the difference vector $\delta x = (\delta q, \delta v)$ between the circular solution and the solution for $\epsilon > 0$.
   \[
   \delta x = \frac{\partial f}{\partial x}(x(t), 0) \delta x + \frac{\partial f}{\partial \epsilon}(x(t), 0) \delta \epsilon(0, 0).
   \]
   Hint: You might want to use Mathematica to solve and simplify this.
   c. For which values of $k$ does the variational ODE give a periodic solution which returns to $\delta x = (0, 0)$ after time $T = 2\pi$. The conclusion is that for this value of $k$, the perturbation somehow cancels out as you go around the circle once. In fact there is a nearby periodic orbit of the VdP equation.

4. (A Center Manifold) Consider an autonomous system in $\mathbf{R}^2$ of the form
   \[
   \begin{bmatrix}
   \dot{x} \\
   \dot{y}
   \end{bmatrix}
   = \begin{bmatrix}
   0x + g_1(x, y) \\
   \lambda y + g_2(x, y)
   \end{bmatrix}
   \]
   where $\lambda > 0$ and the functions $g_i(x, y)$ are $C^1$ and have the following properties:
i. \( g_i(0, 0) = 0 \) and \( Dg_i(0, 0) = (0, 0) \)

ii. \( g_i(x, y) = 0 \) outside of some ball around the origin

iii. \( g_i \) are Lipschitz with Lipschitz constant \( \epsilon \).

Note that the linearized ODE at the origin has eigenvalues \( 0, \lambda > 0 \). The \( y \)-axis is the one-dimensional unstable subspace, \( E^u \), and the \( x \)-axis is the one-dimensional center subspace, \( E^c \). In this problem you will show that if \( \epsilon \) is sufficiently small there exists a curve \( W^c \) in the form of a graph \( y = h(x) \) of a Lipschitz function, consisting of all orbits for which \( |y(t)| \) remains bounded as \( t \to \infty \).

a. Show that if \( \epsilon \) is sufficiently small, then the family of cones \( V(x, y) = \{(x', y') : |y - y'| \geq |x - x'|\} \) is positively invariant and that if \( (x', y') \in V(x, y) \) and \( (x', y') \neq (x, y) \), then for the corresponding solutions, \( |y(t) - y'(t)| \to \infty \) as \( t \to \infty \).

b. Show that if \( \Gamma \) is a “vertical curve”, i.e., the graph of a Lipschitz function \( x = g(y) \) with Lipschitz constant 1, then for \( t \geq 0 \) \( \phi_t(\Gamma) \) is also a vertical curve. Use this to show that each vertical line \( x = x_0 \) contains exactly one point such that \( |y(t)| \) remains bounded as \( t \to \infty \). Note: \( |x(t)| \) is automatically bounded here since \( \dot{x} = 0 \) outside a neighborhood of the origin. Hence \( |y(t)| \) bounded is equivalent to \( (x(t), y(t)) \) remaining in the horizontal cone \( H(0,0) \).

c. Show that the points found in part b fit together to form the graph of a Lipschitz function \( y = h(x) \) with Lipschitz constant 1.