Math 5535 – Homework VI

Hand in starred problems on Wednesday, December 14 (day of Midterm III).

From the text.

Section Problems
13.1 1*, 3, 5*, 6
13.3 2*
14.3 2, 3, 6*, 7
14.1 1, 2

Monday problem sessions

Date Problems
Dec. 5 Section 13.1, 13.2 and additional problems 1,2,3
Dec. 12 Sections 14.1, 14.3 and additional problem 4,5,6,7

1. Let $\sigma : \Sigma_2 \to \Sigma_2$ be the two-sided shift on two symbols (0 and 1), that is, $\Sigma_2$ is the set of bi-infinite sequences $s = \ldots s_{-2}s_{-1}.s_0s_1 \ldots$ with the distance function $d(s, t) = \sum_{n=-\infty}^{\infty} \frac{\delta(s_n, t_n)}{3^n}$ with $\delta(0, 0) = \delta(1, 1) = 0$ and $\delta(1, 0) = \delta(0, 1) = 1$.

a. Find the distance between the sequences $s = \bar{0}.\bar{0}$ and $t = \bar{0}.\bar{1}.\bar{0}$. 

b. Show that periodic points are dense in $\Sigma$.

2. Find an iterated function system whose attractor $A$ is the fractal above. What is the similarity dimension of $A$? Hint: $A$ contains three small copies of itself. Find the three similarities which take $A$ onto these copies.
3*. Consider the function system $F = \{F_0, F_1, F_2, F_3\}$ consisting of four similarities $F_i(X) = AX + B_i$ of the plane with the same matrix and different translations:

$$A = \begin{bmatrix} 1 & -\frac{1}{2} \\ \frac{1}{4} & 1 \end{bmatrix} \quad B_0 = \begin{bmatrix} \frac{3}{2} \\ 1 \end{bmatrix} \quad B_1 = \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix} \quad B_2 = \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix} \quad B_3 = \begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}.$$

a. Express each of the maps in complex notation $F_i(z) = cz + d$. Show that each map is a contraction and find their scale factors.

b. Starting from a square initial set $A_0 = [-1, 1] \times [-1, 1]$, sketch $A_0, F(A_0)$ and $F^2(A_0)$.

c. Find the similarity dimension, $\dim_s(A)$, of the attractor.

4*. Find the Hausdorff distance $D(A, B)$ between the following pairs of compact sets:


c. $A = [0, 1], B = K$, the middle-thirds Cantor set in $\mathbb{R}$.

d. $A = \{(x, y) : x^2 + y^2 \leq 1\}, B = \{(x, y) : x^2 + y^2 \leq 4\}$ in $\mathbb{R}^2$.

e. $A = \{(x, y) : 1 \leq x^2 + y^2 \leq 4\}, B = \{(x, y) : x^2 + y^2 \leq 4\}$ in $\mathbb{R}^2$.

5*. Construct a lopsided Cantor set as follows. Starting from $[0, 1]$, remove the interval $(\frac{1}{4}, \frac{1}{2})$ (call this a quarter-interval left-of-center). Then repeat this process of removing a quarter-interval left-of-center from each of the two closed intervals $[0, \frac{1}{4}], \left[\frac{1}{2}, 1\right]$. Repeat infinitely often to obtain a set $A$. $A$ will be self-similar via two similarities $F_0(x), F_1(x)$ with scale factors $\frac{1}{4}, \frac{1}{2}$. Find these similarities and compute $\dim_s(A)$. Hint: in this case you can solve the required equation for the dimension exactly if you use the fact that $\frac{1}{4} = (\frac{1}{2})^2$.

6. Let $K$ be the middle-thirds Cantor set and consider the Cartesian product $K \times K \subset \mathbb{R}^2$, that is, $K \times K = \{(x, y) : x \in K, y \in K\}$. Show that $K \times K$ has topological dimension $\dim_t(K \times K) = 0$. Find an IFS which has $K \times K$ as attractor and find the similarity dimension $\dim_s(K \times K)$.

7*. For $0 < \sigma < 1$, let $K_\sigma$ be the middle-$\sigma$ Cantor set, that is, remove the middle interval of relative length $\sigma$ at each stage. For example, $\sigma = \frac{1}{3}$ gives the standard Cantor set. The sets $K_\sigma$ all have topological dimension 0 and they are all homeomorphic.

a. Find the similarity dimension of $K_\sigma$. Show that this dimension takes all values $0 < d < 1$ as $\sigma$ varies over $0 < \sigma < 1$.

b. Find $\sigma$ such that $\dim_s(K_\sigma) = \frac{1}{2}$.