Math 5535 – Homework II

Hand in the starred problems only on Wednesday, October 5 (the day of Midterm I)

From the text.
Section Problems
9.4 1, 2*, 3
9.5 1*, 2, 3*, 5a
9.6 1, 2*, 3
9.7 1 (see hint), 2*, 3

Monday problem sessions
Date Problems
Sept. 26 Section 9.4, 9.5 and additional problems 1,2
Oct. 3 9.6, 9.7 and additional problem 3

Plus the following additional problems.
1*. Let \( f(x) = ax(1 - x) \). Show that there is a parameter value \( a \in [3.5, 4] \) such that the critical point \( x_0 = \frac{1}{2} \) is eventually fixed in the sense that \( f^3(1/2) = (a - 1)/a \) (recall that \( \bar{x} = \frac{a-1}{a} \) is a fixed point). Use this and Singer’s theorem to show that for this value of \( a \), \( f(x) \) has no periodic attractors of any period.

2*. Consider the quadratic maps \( f(x) = x^2 + c \) where \( c \) is a constant. Draw an accurate bifurcation diagram showing the curves of fixed points and points of minimal period \( q = 2 \) in the \((c, x)\)-plane (they are both sideways parabolas). Indicate attractors with a solid line and repellers with a dashed line, or use different colors for these. Identify the exact coordinates of the saddle-node bifurcation and the period-doubling bifurcations 1 \( \rightarrow \) 2 and 2 \( \rightarrow \) 4.

3.* Let \( f_a(x) = ax(1 - x) \) be the logistic map and \( g_c(y) = y^2 + c \).
   a. Show that for each \( a \neq 0 \) there is a \( c < \frac{1}{4} \) such that \( f_a(x) \) is conjugate to \( g_c(y) \) by a conjugacy of the form \( y = h(x) = kx + l \) where \( k, l \) are constants. Hint: Set up the conjugacy equation \( h(f_a(x)) = g_c(h(x)) \) and solve for the constants.
   b. Find the fixed points and period 2 points for both dynamical systems and verify that your conjugacy maps these points to one another.

Hints:
Section 9.7.1 b. It is easy to handle the parameter range \( 1 < a \leq e \). For the range \( e < a < e^2 \) plot the graph of \( f^2(x) \) to see why it’s true. Proving it would be a challenge though.
Additional problem 1. For the first part you need to show that the equation \( f^3(1/2) = (a - 1)/a \) has a solution in \([3.5, 4] \).