9.1.6 First we show by induction that the \( n \)-th iterate of the tent map is given by formulas of the form \( T^n(x) = \pm 2^n x + k \) on each of the intervals \( I_m = [m/2^n, (m + 1)/2^n) \) of length \( 1/2^n \) in \([0,1)\). It’s true for \( n = 1 \). Assume it’s true for \( n \) and show for \( n + 1 \). Now \( T^{n+1}(x) = T^n(T(x)) \). Let \( J \) denote one of the intervals \([m/2^{n+1}, (m + 1)/2^{n+1})\). If \( J \subset [0, \frac{1}{2}) \) then \( T(x) = 2x \) and \( T^{n+1}(x) = 2^{n+1}x + k \), while if \( J \subset [\frac{1}{2}, 1) \) we need to substitute \( T(x) = 2 - 2x \) which gives \( T^{n+1}(x) = 2^{n+1}(2 - 2x) + k = \mp 2^{n+1}x + (k + 2^{n+1}) \) which again has the required form.

Now for the part about rational numbers. If \( x_0 \) is a rational number of the form \( x_0 = \frac{p}{q} \) then each of the iterates \( x_n \) can also be written in the form \( x_n = \frac{k}{q} \) for some integer \( 0 \leq k \leq q - 1 \). (It might be possible to reduce such a fraction by cancelling a common factor but we don’t have to do that.) Since there are only \( q \) of these fractions, they must eventually repeat. That is, \( x_n = x_k \) for some iterates \( k < n \). This means that \( x_k \) is a periodic point of some period (a divisor of \( n - k \)) and so \( x_0 \) is eventually periodic. For the converse we assume \( x_0 \) is eventually periodic, that is, assume \( T^n(x) = T^k(x) \) for some \( k < n \). Using the formulas proved by induction above, one can simply solve this for \( x \) obtaining a quotient of two integers, as required.

9.2.5 a. Since \( f \) maps \([0,1]\) onto itself, there are points \( x_0, x_1 \in [0,1] \) with \( f(x_0) = 0 \) and \( f(x_1) = 1 \). To find fixed points we want to consider the function \( g(x) = f(x) - x \). We have \( g(x_0) = 0 - x_0 \leq 0 \) and \( g(x_1) = 1 - x_1 \geq 0 \) (we are using the fact that \( x_0 \geq 0 \) and \( x_1 \leq 1 \)). Now as \( x \) runs over the interval \([x_0, x_1]\), the intermediate value theorem says that \( g(x) \) takes on all values between \( g(x_0), g(x_1) \). In particular, the value \( g = 0 \) occurs somewhere and this is a fixed point.

Depending on where the points \( x_0, x_1 \) lie we can sometimes prove the existence of more fixed points. The best case is \( 0 < x_0 < x_1 < 1 \). Then \( g(x_0) < 0 \) and \( g(x_1) > 0 \) whereas \( g(0) \geq 0 \) and \( g(1) \leq 0 \). The we can apply the intermediate value theorem separately on \([0, x_0], [x_0, x_1], [x_1, 1] \). Note that in this case the dividing points \( x_0, x_1 \) are not fixed points so the fixed points must be in \([0, x_0), (x_0, x_1), (x_1, 1]\) and we get at least one fixed point in each interval for a total of at least 3. If we have \( x_0 < x_1 \) but \( x_0 = 0 \) or \( x_1 = 1 \), we might only have two fixed points.

9.2.5 b,c. From the preceding analysis we see that \( f \) itself has at least 2 fixed points provided the points \( x_0, x_1 \) are in the order \( x_0 < x_1 \) and at least 3 if \( 0, 1 \) are not fixed. These would also be fixed by \( f^2(x) \). It remains to show that if the points are in the order \( x_1 < x_0 \) then the second iterate has points in the order \( y_0 < y_1 \) with \( f^2(y_0) = 0, f^2(y_1) = 1 \). Since \( x_1 < x_0 \) the map \( f \) maps the interval \([x_1, x_0]\) onto \([0,1]\) and in “reverse order”, i.e., with \( f(x_1) = 1, f(x_0) = 0 \). We will look for points \( y_0 < y_1 \) in \([x_1, x_0]\) which map onto \( x_0, x_1 \). By the intermediate value theorem, there is some \( y_0 \in [x_1, x_0] \) with \( f(y_0) = x_0 \) and so \( f^2(y_0) = 0 \). Now consider the interval \([y_0, x_0]\) which gets mapped onto \([0, x_0]\) (in reverse). Since \( x_1 \) is in this interval, there is some \( y_1 \in [y_0, x_0] \) with \( f(y_1) = x_1 \) and so \( f^2(y_1) = 1 \). Since \( y_1 \) cannot be equal to \( y_0 \), we have \( y_0 < y_1 \) as required. Apply the reasoning from part a to \( f^2 \) to get b and c.
Extra problem 1. One can check that \( f(1) - 1 > 0 \) and \( f(2) - 2 < 0 \) so there is a fixed point \( \bar{x} \in [1, 2] \). Here are two approaches to the problem of global attraction. The first uses distance contraction and the other uses graphical analysis (cobweb plots).

First note that for any \( x_0 \), the first iterate \( x_1 = f(x_0) \in [0, 2] \) and then the second iterate \( x_2 \in [1, 2] \). So it suffices to show that every point in \([1, 2]\) gets attracted. We will use the fact that the map on this interval contracts distances. The derivative satisfies \( |f'(x)| = \text{sech}^2(x) = \frac{4}{(e^x + e^{-x})^2} \leq 1 \) but it is equal to 1 when \( x = 0 \) so we do not immediately get a distance-contraction factor \( \lambda < 1 \). But we can get such a factor on any interval which avoids \( x = 0 \), for example, consider \( I = [1, 2] \). Looking at the graph of \( f \) it is clear that for any \( x \geq 0 \), we have \( 1 \leq f(x) < 2 \). In particular, \( f \) maps \( I \) into itself, \( f: I \to I \). And on \( I \) we have \( |f'(x)| \leq \text{sech}^2(1) < 1 \) so by the mean value theorem, \( f: I \to I \) contracts distances by at least \( \lambda = \text{sech}^2(1) \). It follows at for \( x_0 \in [1, 2] \) the distance to the fixed point satisfies \( d(f^n(x_0), \bar{x}) \leq \lambda^n d(x_0, \bar{x}) \). So these orbits are attracted to \( \bar{x} \).

Here is the graphical analysis version. Note that the derivative satisfies \( 0 < f'(x) \leq 1 \) and is strictly less than 1 away from the origin. So \( f(x) \) is strictly increasing. It covers the range \( 0 < f(x) < 2 \) as \( x \) runs from \(-\infty \) to \( \infty \). The value at \( x = 0 \) is \( f(0) = 1 \). Since the slope is strictly less than 1 except at \( x = 0 \), the function \( f(x) - x \) is strictly decreasing and it follows that there can be only one fixed point, the point \( \bar{x} \in [1, 2] \) mentioned above. It follows from this analysis that on the interval \( (-\infty, \bar{x}) \) we have \( x < f(x) < \bar{x} \) and on the interval \( (\bar{x}, \infty) \) we have \( \bar{x} < f(x) < x \). A theorem in the text shows that the basin of attraction contains both of these intervals, i.e., it’s the whole real line. As a review, here is the reason why this works. If \( x_0 \in (-\infty, \bar{x}) \), the orbit \( x_n = f^n(x_0) \) remains in this interval for all \( n \) and the sequence is monotonically increasing. Since it is bounded above by \( \bar{x} \) it has a limit which must be a fixed point and since \( \bar{x} \) is the only fixed point, we have \( x_n \to \bar{x} \). Similarly, if \( x_0 \in (\bar{x}, \infty) \), the orbit \( x_n = f^n(x_0) \) remains in this interval for all \( n \), the sequence is monotonically decreasing and must converge to \( \bar{x} \).

Extra problem 2. For any odd function, \( f(-x) = -f(x) \). Setting \( x = 0 \) gives \( f(0) = -f(0) \) which implies \( f(0) = 0 \). So 0 is a fixed point. If \( x_0 \neq 0 \) and \( f(x_0) = -x_0 \) then \( f^2(x_0) = f(f(x_0)) = f(-x_0) = -f(x_0) = -(x_0) = x_0 \) so \( x_0 \) is periodic of period 1 or 2. It’s not fixed since \( f(x_0) = -x_0 \neq x_0 \) so it has (minimal) period 2. Now consider \( q \geq 2 \). The \( q \)-th iterate \( f^q(x) \) is also an odd function (as is easy to show by induction). So the same argument shows that if \( f^q(x_0) = -x_0 \) then \( f^{2q}(x_0) = x_0 \) and that \( x_0 \) is not fixed by \( f^q \). To see that \( 2q \) is the (minimal) period suppose that there is some \( k < 2q \) such that \( f^k(x_0) = x_0 \). \( k \) must divide \( 2q \) but it can’t divide \( q \). The only other possibility is that \( k \) is an even number of the form \( k = 2l \) where \( l < q \) is a divisor of \( q \). Then \( q = m \cdot l \) where \( m > 1 \). So we have \( f^{2l}(x_0) = x_0 \) and \( f^{ml}(x_0) = -x_0 \). It follows that

\[-x_0 = f^{ml}(x_0) = f^{(m-2)l}(f^{2l}(x_0)) = f^{(m-2)l}(x_0).\]

This contradicts the fact the \( q = ml \) was the smallest exponent with \( f^q(x_0) = -x_0 \).