Math 4242 – More problems on quadratic forms

1. For each of the following symmetric matrices, $A$, find an orthogonal matrix $P$ such that $P^TAP$ is diagonal.

   a. $\begin{bmatrix} -3 & 4 \\ 4 & 3 \end{bmatrix}$  
   b. $\begin{bmatrix} 2 & -1 \\ -1 & 4 \end{bmatrix}$  
   c. $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

2. Check that the following matrix, $A$, is Hermitian and find a unitary matrix $P$ such that $P^*AP$ is diagonal.

   $\begin{bmatrix} 1 & i \\ -i & 0 \end{bmatrix}$

3. Write each of the following quadratic forms as $Q(X) = X^TAX$ where $A$ is a symmetric matrix. Then find an orthogonal change of variables $X = PY$ such that $Q(Y) = Y^TDY$ and $D$ is diagonal.

   a. $Q(X) = 2x_1^2 - 2x_1x_2 + 4x_2^2$  
   b. $Q(X) = x_1x_2$  
   c. $Q(X) = x_1^2 + 2x_1^2 + x_2^2 + 2x_1x_2 + 2x_2x_3$

4. Sketch the curves $Q(X) = 1$ for the quadratic forms in problems 3a and 3b. If the curve is an ellipse find the lengths of the semi-major axes.

5. Show that the eigenvalues of an antisymmetric matrix $A$ are purely imaginary numbers, $\lambda = ki$ where $k$ is real. If $V, W$ are eigenvectors for eigenvalues $\lambda, \mu$ with $\lambda + \mu \neq 0$ then $V, W$ are orthogonal. Hint: first show that for any two vectors $V, W$ we have $\langle V|AW \rangle = -\langle AV|W \rangle$ where $\langle V|W \rangle = V^TW$.

6. Suppose $A$ is a symmetric matrix and $Q(X) = X^TAX = 1$ is an ellipse. How are the semi-major axes related to the singular values of $A$?

7. For each of the following matrices find the polar factorization $A = RU$ (see problem 7.6.7).

   a. $\begin{bmatrix} 3 & 4 \\ 5 & 0 \end{bmatrix}$  
   b. $\begin{bmatrix} 2 & -1 \\ -i & 2i \end{bmatrix}$