Certain subsets of $\mathbb{R}^n$ will be easiest to deal with, and arise frequently:

**DEFINITION:** A (vector or linear) subspace $V \subset \mathbb{R}^n$ is a subset which is

(a) nonempty $V \neq \emptyset$

(b) closed under $+$, that is, $\vec{v}, \vec{w} \in V \Rightarrow \vec{v} + \vec{w} \in V$

(c) closed under scalar multiplication, that is, $c \vec{v} \in V$, $c \in \mathbb{R} \Rightarrow c \vec{v} \in V$

(5.1)

$$\begin{align*}
\text{(In particular, (a),(c) \Rightarrow \ \vec{0} = 0 \cdot \vec{v} \quad \forall \vec{v} \in V)} \\
\text{so } V \text{ contains the origin}
\end{align*}$$

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**EXAMPLES:**

1. $\{0\}$, $\mathbb{R}^n$ are trivial subspaces inside $\mathbb{R}^n$

2. Lines, planes through the origin

3. See EXERCISE 1.1.9 $\implies$ got discussed a bit already in recitation; for fixed $\vec{w} \in \mathbb{C}$, $\{z \in \mathbb{C} : \text{Re}(zw) = 0\}$

**NON-EXAMPLES:**

4. Lines, planes not through the origin (Why not?)

5. Union of two lines through origin (Why not?)

6. Unit circle $x^2 + y^2 = 1$ (Why not?) (Lots of non-examples!)
REM: Later in the course, we encounter a situation where distinguishing between points \((x, y)\) versus vectors \([x, y]\) makes a lot of sense.

**DEF'N:** A vector field on \(\mathbb{R}^n\) is a function

\[
f: \mathbb{R}^n \rightarrow \mathbb{R}^n
\]

sending a point \((x_1, \ldots, x_n)\) to \([y_1, \ldots, y_n] = f(x_1, \ldots, x_n)\), a vector.

**EXAMPLES** in \(\mathbb{R}^2\):

1. \(f(x, y) = [x, y]\) the radial vector field

   ![](radial_vector_field.png)

2. \(f(x, y) = [-y, x]\)

   ![](circular_vector_field.png)

Later we calculate **line integrals for curves through vector fields**, giving work done, or electrical flux, or one can solve differential equations to find flow lines through the vector field.
§1.2, 1.3 Matrices and Linear Transformations

**Def:** The easiest functions $f: \mathbb{R} \rightarrow \mathbb{R}$ to graph or solve equations $f(x) = c$

are ones of the form $f(x) = mx + b$

and we use them to approximate more complicated functions $g(x)$

by taking derivatives:

A different way to say this:
Translate the graph of $y=g(x)$ so that $(x_0, y_0)$
on it goes through $B$ by replacing
$g(x)$ with $g(x+x_0) - y_0 = g(x)$

The interesting part of $f(x) = mx + b$ is the $f(x) = mx$ (the linear part);
then $f(x) = T(x) + b$ is just adding on a fixed vector/point $b$.

Let's generalize $f(x) = mx$ to $\mathbb{R}^n$ ...

**Def:** A function $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$ is called a linear transformation
if it respects vector $+$ and scalar multiplication,

i.e. $T(\mathbf{v} + \mathbf{w}) = T(\mathbf{v}) + T(\mathbf{w})$ \quad $\forall \mathbf{v}, \mathbf{w} \in \mathbb{R}^n$

$T(cv) = cT(\mathbf{v})$ \quad $\forall c \in \mathbb{R}$

$\Rightarrow T(c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 + \ldots + c_k \mathbf{v}_k) = c_1 T(\mathbf{v}_1) + \ldots + c_k T(\mathbf{v}_k)$

$\Rightarrow T(\frac{1}{i = 1} \sum c_i \mathbf{v}_i) = \frac{1}{i = 1} \sum c_i T(\mathbf{v}_i)$

i.e. $T$ respects linear combinations
EXAMPLES:

1. Scalings
   \[ T(x) = c \cdot x = \begin{bmatrix} cx_1 \\ \vdots \\ cx_n \end{bmatrix}, \quad T: \mathbb{R}^n \rightarrow \mathbb{R}^n \]
   \[ T(c + \vec{w}) = c \cdot (c + \vec{w}) = c \cdot \vec{v} + c \cdot \vec{w} \]

2. Rotations through some angle \( \theta \) about \( \vec{0} \) in \( \mathbb{R}^2 \)

3. Reflections in a line through \( \vec{0} \) in \( \mathbb{R}^2 \)
   or a plane through \( \vec{0} \) in \( \mathbb{R}^3 \)

4. Projection orthogonally onto a line through \( \vec{0} \) in \( \mathbb{R}^2 \)
   (perpendicularly) or a plane through \( \vec{0} \) in \( \mathbb{R}^3 \)

NON-EXAMPLES:

5. Rotations not about \( \vec{0} \), about some \( \vec{x} \neq \vec{0} \), in \( \mathbb{R}^2 \)

6. Reflections in lines not through \( \vec{0} \) in \( \mathbb{R}^2 \)

7. Projections orthogonally onto lines not through \( \vec{0} \)

8. \( f: \mathbb{R} \rightarrow \mathbb{R} \) given by \( f(x) = x^2 \), since \( (x+y)^2 \neq x^2 + y^2 \)

9. \( f(x) = \sin(x) \), since \( \sin(x+y) \neq \sin(x) + \sin(y) \)

10. \( f(x) = mx + b \) with \( b \neq 0 \)