09/23/2016  Limits and Continuity

- Open sets, closed sets, etc...

We start with

define open ball $B_{e}(x) = \{ y \in \mathbb{R}^n \mid \| y - x \| < e \}$

defn: A set $U \subseteq \mathbb{R}^n$ is called open if $\forall x \in U,$

\[ \exists \, \varepsilon > 0 \text{ s.t. } B_{e}(x) \subseteq U \]

"for every pt in $U$, there is an open ball in $U$ centered at that pt."

Examples:

1) $(a, b) \subseteq \mathbb{R}$: 

$$ \left( \frac{e}{2}, \frac{e}{2} \right) $$

i.e. $e = \min \left\{ \frac{b-a}{2}, \frac{b-x}{2} \right\}$

2) generic:

"boundary pt not in $U$"

3) non-example $(a, b) \subseteq \mathbb{R}^2$:

$$ a \ x \ b $$

$$ e $$
Define: Closed sets: A set \( C \subseteq \mathbb{R}^n \) is closed if its complement \( \mathbb{R}^n - C \) is open.

Examples:

1. Square with inner part:
   - Indeed, complement is open.
   - Each pt \( x \) has minimum distance from square.

2. Empty square.
   - Claimed.

iii) Non-example:

\[ A = \{ 0, \frac{1}{n}, \ldots \} \]

Here \( 0 \in A^c \), but for any \( \varepsilon > 0 \), \( B(0, \varepsilon) \cap A = \emptyset \) i.e. \( B(0, \varepsilon) \not\subseteq A^c \) so \( A^c \) is not open, so \( A \) is not closed.

- Enough to show that if \( \varepsilon > 0 \), \( \exists n \in \mathbb{N} \) numbers e. t. h., \( \frac{1}{n} < \varepsilon \). Picking \( n > \frac{1}{\varepsilon} \) (Archimedean property)
Naturally appearing set:
\[ y > 0 \Rightarrow \frac{x}{y} > 0 \Rightarrow \text{not in domain} \]

Example 1.5.5: \( f(x,y) = \sqrt{\frac{x}{y}} \)

We need \( y \neq 0, \frac{x}{y} > 0 \Rightarrow \)

\( x = \text{axis}\) is not in domain.

A naturally occurring set
is neither open nor closed.
Look at two axes. \( \Rightarrow \)

\( \Rightarrow \) "Sets are not closed."\( \Rightarrow \)

\( \emptyset \subset \mathbb{Q} \): Example of a set that is
both open and closed? \( \Rightarrow \mathbb{R} \)

Why is \( \emptyset \) open? \( \Rightarrow \) "vacuous truth"

Rational numbers: \( \mathbb{Q} \subset \mathbb{C} \Rightarrow \mathbb{R} \)

Is \( \emptyset \) open? No; any \((a,b)\) contains irrationals.

Is \( \emptyset \) closed? No; any \((a,b)\) contains irrationals.

However \( \mathbb{Z} \subset \mathbb{R} \Rightarrow \text{closed} \).

Intuitive sets of points

\[ -2 -1 0 \quad \mathbb{Q} \quad 1 \quad 2 \]

\( \text{can be closed.} \)
Q: An open set \( U \subset \mathbb{R} \) s.t. \( Q \subset U \) and \( U \neq \mathbb{R} \)

A: \( \mathbb{R} \times \{ n \pi \} \) any irrational

Q: An open set \( U \subset \mathbb{R} \) s.t. \( Q \subset U \) and \( U \) has "finite length"?

A: use balls with radius \( \frac{1}{2^n} \) (where \( \sum \frac{1}{2^n} = 2 \))

Need: Rational numbers are countable...

Three more concepts:

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<th>Term</th>
<th>Idea</th>
<th>Definition</th>
<th>Example</th>
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<tr>
<td>closure:</td>
<td>The smallest ( \overline{U} = { x \in \mathbb{R}^n \mid B_r(x) \subset U \neq \emptyset } ) if ( r &gt; 0 )</td>
<td>( \square \rightarrow \square )</td>
<td></td>
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<tr>
<td>interior:</td>
<td>The largest ( U^o = { x \in \mathbb{R}^n \mid \forall r &gt; 0 \text{ with } B_r(x) \subset U } )</td>
<td>( \square \rightarrow \square )</td>
<td></td>
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<tr>
<td>boundary:</td>
<td>Those pts that might be added/removed when taking closure/interior for all ( r &gt; 0 )</td>
<td>( \emptyset ) inside ( \square )</td>
<td></td>
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"Complement"
Limits of sequences:

Defn: A sequence \( \{a_n\}_{n=1}^{\infty} \) converges to a point \( \alpha \in \mathbb{R}^k \) if for every \( \varepsilon > 0 \) there is some \( M \) s.t. \( n > M \) \( |a_n - \alpha| < \varepsilon \).

(For a challenge \( \varepsilon \), I have to provide an answer \( M \)).

Examples: 1) \( a_n = \frac{1}{n} \in \mathbb{R}^1 \): \( a_n \to 0 \).

I need to show that for all \( \varepsilon > 0 \), \( \exists M \) s.t. for all \( n > M \) I have \( |\frac{1}{n} - 0| < \varepsilon \). Pick any \( M > \frac{1}{\varepsilon} \).

=> Rethink our example: \( \{\frac{1}{n}\} \) was not a closed set because it did not contain all of its limit points.

2) \( a_n = \left( \frac{1}{n^2} \right) \in \mathbb{R}^2 \). \( a_n \to (0) \) clearly. But how to prove?

\[ |a_n - (0)| < \varepsilon \iff \sqrt{\frac{1}{n^2} + \frac{1}{(n+1)^2}} < \varepsilon \]

Some not so difficult algebra should work:

\[ \sqrt{\frac{1}{n^2} + \frac{1}{(n+1)^2}} < \sqrt{\frac{2}{n^2}} = \frac{\sqrt{2}}{n} \to \text{to be } < \varepsilon. \]
Better way: I would like theorem to say: 
\[ \frac{1}{n} \to 0 \Rightarrow \frac{n}{n+1} \to 1 \Rightarrow \left( \frac{n}{n+1} \right) \to 1 \]

This is true indeed:

If \( \{a^n\} \subseteq \mathbb{R}^k \) and \( a_n = (a_{n1}, \ldots, a_{nk}) \) and \( a = (a_1, \ldots, a_k) \), I am given that \( (a_{n1}) \to a_1 \) and I want to show that \( (a_{nk}) \to a_k \).

The assumption means that for any given \( \varepsilon_1, \ldots, \varepsilon_k \), I can find \( M_1, \ldots, M_k \) so that

\[ |(a_{n1}) - a_1| < \varepsilon_1 \quad \forall n > M_1 \]
\[ |(a_{n2}) - a_2| < \varepsilon_2 \quad \forall n > M_2 \]
\[ \cdots \]
\[ |(a_{nk}) - a_k| < \varepsilon_k \quad \forall n > M_k. \]

To show that \( (\mathbf{a}_n) \to \mathbf{a} \), I want for any \( \varepsilon > 0 \) an \( M \) s.t. \( n > M \)

\[ |\mathbf{a}_n - \mathbf{a}| < \varepsilon \quad \Rightarrow \sqrt{[a_{n1} - a_1]^2 + \cdots + [a_{nk} - a_k]^2} < \varepsilon \]

Choose \( M > \max \{M_1, \ldots, M_k\} \) and you have

\[ \sqrt{\varepsilon_1^2 + \cdots + \varepsilon_k^2} < \varepsilon. \]
That is, for a given $\epsilon > 0$, I have to find\ indica small positive $\epsilon_i$'s such that
\[
\sqrt{\epsilon_1^2 + \ldots + \epsilon_k^2} < \epsilon
\]
This is definitely doable, so pick $\epsilon_i < \frac{\epsilon}{\sqrt{k}}$ for example.