Proof (roughly D’Alembert’s)  

1746 proof  

We’d like to say: the root \( z_0 \) is the \( z_0 \in \mathbb{C} \) achieving the minimum value \( f(z_0) = 0 \) where \( f(z) = |p(z)| \) i.e.: \( \mathbb{C} \xrightarrow{p} \mathbb{C} \xrightarrow{z \mapsto p(z)} \mathbb{R} \)

Does \( f: \mathbb{C} \rightarrow \mathbb{R} \) achieve a minimum?  

Certainly, \( f \) is continuous (Why?)  

But \( \mathbb{C} \) is not compact (This is a problem for \( \frac{1}{1+z^2} \) and for \( e^z \))

We claim that \( f(z) = |p(z)| \) should achieve a minimum for some choice of \( R \), which is compact (by Extreme Value Theorem)

and this should be its global minimum \( f(z_0) \)

because \( f(z) = |p(z)| = |z^k + a_{k-1}z^{k-1} + \ldots + a_1z + a_0| \geq |a_0| \)

\( f(c_0) \)

for \( R \) sufficiently large. How large?  

triangle inequality

If \( |z| > R \) then \( f(z) = |p(z)| \geq |z^k| - |a_{k-1}z^{k-1} + \ldots + a_1z + a_0| \)

\( \geq R^k \)

(should dominate for \( R \) large)

\( \leq |a_{k-1}| |z^{k-1}| + \ldots + |a_1||z| + |a_0| \)

\( \leq k \cdot \max\{|a_{k-1}|, \ldots, |a_0|\} \cdot |z|^{k-1} \)

\( A = \)

\[ f(z) \geq R^k - kAR^{k-1} \]

\[ = R^{k-1}(R - kA) \]

\( \geq R^{k-1}A \) if \( R \geq (k+1)A \)

\( \geq A \) if \( R \geq 1 \)

\( \geq |a_0| \) if we choose \( R = \max\{|a_{k-1}|, \ldots, |a_0|\} \)

Now since \( z_0 \) achieves minimum of \( |p(z)| \), we want to show \( |p(z_0)| = 0 \) i.e. \( p(z_0) = 0 \). If not, so \( |p(z_0)| > 0 \), we'll show \( f \) some \( z_1 \) in a small circle around \( z_0 \) with \( |p(z_1)| < |p(z_0)| \).
The algebra is easier if we replace \( p(z) \) by \( q(z) = p(z + z_0) \)
(i.e. \( p(z) = q(z - z_0) \))
which has same values for \( |q(z)| \), so still has minimum value \( |q(z_0)| = |p(z_0)| \), and \( q(z) \) is still a degree-k polynomial
since \( q(z) = p(z + z_0) = (z + z_0)^k + q_{k-1}(z + z_0)^{k-1} + \ldots + q_1(z + z_0) + q_0 \)
\[ = z^k + b_{k-1}z^{k-1} + \ldots + b_1 z + b_0 \]
\[ b_0 = q(z_0) = p(z_0) \neq 0 \text{ by assumption} \]

Now write \( q(z) = b_0 + b_1 z + \ldots + b_k z^k \)

A key point (again see footnote):

- \( \theta = x + iy \) has \( z = e^{i\theta} \)
- so as \( \theta \) goes from \( 0 \) to \( 2\pi \)
  - \( z \) travels once around,
  - \( \arg z \) orbits once around

When \( |z| = \rho \) is small,
\[ b_0 + b_j z^j \] travels in a small circle around \( b_j \) times.

We'll try to find \( z_1 \) on this circle making \( |q(z_1)| < |q(z_0)| \)
with \( |z_1| = \rho \)

Pick \( \epsilon \) very small, and pick \( z_1 \) as in this picture:

Then we'll have \( |q(z_1)| < |b_0| = |q(z_0)| \)
as long as \( |b_{j+1} z^{j+1} + \ldots + b_{k-1} z^{k-1} + z^k| < |b_j| \rho^j \)
\[ \leq |b_{j+1}| \rho^{j+1} + \ldots + |b_{k-1}| \rho^{k-1} + \rho^k \]
\[ \leq \max \{ |b_{j+1}|, \ldots, |b_{k-1}|, 1 \} \cdot (k-j) \rho^{j+1} \]
So we need \( B(k-j) \rho^{j+1} < |b_j| \rho^j \), i.e. \( B(k-j) \rho < |b_j| \) or \( \rho < \frac{|b_j|}{B(k-j)} \)
(46)

Q: Where did this proof fail for \( f(x) = x^2 + 1 \) having no roots \( x \in \mathbb{R} \)?

It does achieve a minimum value \( f(0) = 1 \) at \( x = 0 \).
But the "man" can't walk around the "flagpole" in a full circle, only at 2 points.

Cor 1.6.14: A polynomial \( z^k + a_k z^{k-1} + \ldots + a_1 z + a_0 \) with \( a_i \in \mathbb{C} \) and \( k \geq 1 \)
has exactly \( k \) roots \( r_1, \ldots, r_k \) in \( \mathbb{C} \) (if you count with multiplicity),
and factors as \( p(z) = (z-r_1)(z-r_2) \cdots (z-r_k) \).

Proof: Induct on \( k \). The base case \( k = 1 \) has \( p(z) = z^1 + a_0 = z - r_1 \) with \( r_1 = a_0 \).

In the inductive step, assume it for \( k-1 \), and
given \( p(z) \) of degree \( k \), find some root \( r_k \) using FundThmAlg.

Use long division algorithm to write

\[
\begin{align*}
\overline{z - r_k} & \quad z^k + a_k z^{k-1} + \ldots + a_1 z + a_0 = p(z) \\
\vdots & \\
\vdots & \\
b & \text{remainder}
\end{align*}
\]

However \( r_k \) a root of \( p(z) \) forces

\[
0 = p(r_k) = g(r_k) (r_k - r_1) + b = b, \quad \text{i.e. } p(z) = g(z) (z - r_1)
\]

Now apply induction to \( g(z) \)

What about irreducible factors of \( p(x) = x^k + a_k x^{k-1} + \ldots + a_0 \) with \( a_i \in \mathbb{R} \)
if we only allow real coefficients in the factors?

E.g. \( x^4 - 1 = (x^2 - 1)(x^2 + 1) \)

\[
\begin{align*}
\overline{(x - 1)} & \quad (x - 1)(x + 1)(x^2 + 1) \quad \text{irreducible over } \mathbb{R} \\
\overline{(x + 1)} & \quad (x - 1)(x + 1)(x - i)(x + i) \quad \text{over } \mathbb{C}
\end{align*}
\]