Once one introduces this notion...

**Defn:** A subset $C \subseteq \mathbb{R}^n$ is compact if it is closed and bounded (i.e., $\exists R > 0$ with $B_R(0) \subseteq C$)

One can use what we've learned to prove...

**THM (Bolzano-Weierstrass)**

1.6.3

Every sequence $x_1, x_2, \ldots \subseteq C$ of a compact set in $\mathbb{R}^n$ has a convergent subsequence $(x_{i_k})$ whose limit is in $C$.

**THM (Extreme Value Thm)**

1.6.9

If $C \subseteq \mathbb{R}$ is compact, and $f: C \to \mathbb{R}$ continuous, then $f$ achieves a minimum and maximum value on $C$.

i.e. $\exists a, \, b \in C$ with $f(a) \leq f(x) \leq f(b) \forall x \in C$.

**THM (Mean Value Thm)**

1.6.12

$f: [a, b] \to \mathbb{R}$ continuous and $f$ differentiable on $(a, b)$

$\Rightarrow \exists c \in (a, b)$ with

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

**THM (Fundamental Thm of Algebra)**

1.6.13

Every polynomial $p(z) = z^k + a_{k-1} z^{k-1} + \ldots + a_1 z + a_0$ with $k \geq 1$ has at least one root $z \in \mathbb{C}$.

i.e. $p(z_0) = 0$.

Not at all obvious, and very important!
Theorem (Bolzano-Weierstrass): \((x_i)_{i=1}^{\infty} \subseteq C \subseteq \mathbb{R}^n \Rightarrow \exists \text{ a convergent subsequence } (x_{i_j})_{j=1}^{\infty}

proof: Since \(C\) is bounded, every \(x_i\) has all coordinates in \([-10^m, 10^m]\)
for some \(m\), so \((x_i)_{i=1}^{\infty} \subseteq B\) for some large box \(B\).

E.g., \(n=2\), \(m=3\):

\[
\begin{array}{ccc}
0 & 1 & 2 \\
0 & 1 & 2 \\
0 & 1 & 2 \\
\end{array}
\]

Dividing each coordinate interval \([-10^m, 10^m]\) into 20 equal subintervals
divides \(B\) into \(20^n\) subboxes, at least one of which, call it \(B_1\), has \(x_i \in B\) for infinitely many \(i\):

\[
\begin{array}{ccc}
0 & 1 & 2 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
\end{array}
\]

Pick any \(i\) with \(x_i \in B_1\) and call this \(i(1) = i\).
Repeat this procedure, replacing \(B\) with \(B_1\):

\[
(x_i)_{i=1}^{\infty} \text{ with } (x_i)_{i=1}^{\infty} \subseteq B_1 \cap B_1
\]

producing a subbox \(B_2 \subseteq B_1\) and \(i(2) > i(1)\), with \(x_i(2) \in B_2\),
\(B_3 \subseteq B_2\), \(i(3) > i(2)\), \(x_i(3) \in B_3\),

\[
\begin{array}{ccc}
0 & 1 & 2 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
\end{array}
\]

We claim \((x_i(1), x_i(2), ...)\) is a convergent subsequence:

Every \(x = [x_1, x_2, \ldots] \in B_1\) has the same \(10^m\) decimal digits \([d_1^{(1)}, d_2^{(1)}]\) for their entries

\[
\begin{array}{ccc}
\ldots & \ldots & \ldots \\
1 & 0 & 1 \\
\ldots & \ldots & \ldots \\
\end{array}
\]

so if one defines \(\bar{a} := [d_1, d_2, \ldots]\) by the decimal expansion of its coordinate,

then it's easy to check \(\lim_{j \to \infty} x_{i(j)} = \bar{a}\), since \(|x_{i(j)} - \bar{a}| \leq \sqrt{\sum_{l=1}^{10^{m-j}} (10^{-l})^2}

\[
\leq \sqrt{10^{-m-j}} \to 0 \text{ as } j \to \infty.
\]

Also \(\bar{a} \in C\) since \(C\) is closed.
REMARK: This proof is highly non-constructive: even if we specify a concrete sequence \((x_m)_{m=1}^\infty \subseteq C = [-1,1] \) (in \(\mathbb{R}^1\)), we have no idea what the sequence of subintervals \([0,1] > B_1 \supset B_2 \supset \ldots\) will look like, and how to describe explicitly a convergent subsequence!

\[\text{(Extreme Value Thm)}
\]

**Theorem 1.6.9:** For \( f: C \to \mathbb{R} \) continuous with \( C \) compact,

\[\mathbb{R}^n \ni \exists a, b \in C \text{ with } f(a) \leq f(x) \leq f(b) \text{ for all } x \in C,
\]

(i.e. \( f \) achieves a minimum, maximum value on \( C \))

**Proof:** Let's do max; then applying it to \(-f(x)\) gives the min.

First show the values \( f(x) \) are bounded. If not,
then \( \forall N = 1, 2, \ldots \ \exists x_N \in C \text{ with } f(x_N) > N \).

Use Bolzano-Weierstrass to find a convergent subsequence \((x_{N_j})_{j=1}^\infty \subseteq C \) with \( \lim_{j \to \infty} x_{N_j} = x_0 \in C \)

Continuity implies \( \lim_{j \to \infty} f(x_{N_j}) = f(x_0) \).

This leads to a contradiction: for \( j > f(x_0) + 1 \), one has \( f(x_{N_j}) > N_j \); hence \( f(x_0) > f(x_{N_j}) \).

But if we pick \( \varepsilon > 0 \) with \( 1 > \varepsilon > 0 \) then \( \exists J \) such that \( f(x_{N_j}) - f(x_0) < \varepsilon < 1 \)

\[\Rightarrow f(x_{N_j}) < f(x_0) + \varepsilon < f(x_0) + 1.
\]

When \( j > \max\{f(x_0) + 1, J\} \), these are in conflict.

Once the values of \( f(x) \) are bounded, we know they have a supremum \( M \) in \( \mathbb{R} \):

\[\text{least upper-bound, i.e. } M \geq f(x) \forall x \in C \]but no \( M' < M \) has this property.

But then \( \exists x_1, x_2, \ldots \in C \) with \( \lim_{i \to \infty} f(x_i) = M \) (possibly \( x_1 = x_2 = \ldots \in C \) and \( f(x_i) = M \), so \( \exists \) a convergent subsequence \((x_{i_j})_{j=1}^\infty \) with \( \lim_{j \to \infty} x_{i_j} = \alpha \in C \)

and continuity gives \( M = \lim_{j \to \infty} f(x_{i_j}) = f(x) \).