Q: Where did the proof fail for \( f(x) = x^2 + 1 \) having no roots \( x \in \mathbb{R} \)?

\[ y = x^2 + 1 \]

It does achieve a minimum value \( f(x) = 1 \) at \( x = 0 \).
But the "man" cannot walk around the "flagpole" in a full circle, only at 2 points.

**COR 1.6.14**: A polynomial \( P(x) = a_k x^k + \cdots + a_1 x + a_0 \) with \( a_i \in \mathbb{C} \) and \( k \geq 1 \)
has exactly \( k \) roots \( r_1, \ldots, r_k \) in \( \mathbb{C} \) (if you count with multiplicity),
and factors as \( P(x) = (x - r_1)(x - r_2) \cdots (x - r_k) \).

**Proof**: Induct on \( k \). The base case \( k = 1 \) has \( P(x) = x^1 + a_0 = x - r_1 \) with \( r_1 = a_0 \).

In the inductive step, assume it for \( k - 1 \), and given \( P(x) \) of degree \( k \), find some root \( r_1 \) using Fundamental Thm Alg.

Use long division algorithm to write

\[
\begin{align*}
\frac{P(x)}{x - r_1} &= \frac{a_k x^k + \cdots + a_1 x + a_0}{x - r_1} \\
&= q(x)(x - r_1) + b \quad \text{for some } b \in \mathbb{C} \\
&\quad \text{and polynomial } q(x) \text{ of degree } k - 1, \\
&\quad \text{monic, i.e.} \\
&\quad q(x) = x^{k-1} + b_{k-2} x^{k-2} + \cdots \\
&\quad \text{remainder } b \text{ is of degree } 0, \\
&\quad \text{i.e. } b \in \mathbb{C}.
\end{align*}
\]

However \( r_1 \) a root of \( P(x) \) forces

\[ 0 = P(r_1) = q(r_1)(r_1 - r_1) + b = b \quad \text{i.e. } P(x) = q(x)(x - r_1) \]

Now apply induction to \( q(x) \)

What about irreducible factors of \( p(x) = x^4 + a_k x^4 + \cdots + a_0 \) with \( a_i \in \mathbb{R} \)
if we only allow real coefficients in the factors?

\[ x^4 - 1 = (x^2 - 1)(x^2 + 1) \]

\[ \uparrow \quad \text{or} \quad \downarrow \]

\[ \begin{array}{c}
\psi + i \\
\rightarrow \mathbb{C}
\end{array} \]

\[ \begin{array}{c}
\psi - i \\
\rightarrow \mathbb{R}
\end{array} \]

\[ (x - 1)(x + 1)(x - i)(x + i) \text{ over } \mathbb{C} \]
**Cor 1.6.15:** If \( p(x) = x^k + a_1 x^{k-1} + \ldots + a_k x + a_0 \) with \( a_i \in \mathbb{R} \), \( k \geq 1 \)

then it can be factored \( p(x) = (x-r_1) \ldots (x-r_k)(x^2 + q_1 x + d_1) \ldots (x^2 + q_m x + d_m) \)

for some \( r_1, \ldots, r_k \in \mathbb{R} \)

with \( c_1, \ldots, c_m \)

\( d_1, \ldots, d_m \)

having \( 2m + k = k \) and each quadratic \( x^2 + q_i x + d_i \) irreducible over \( \mathbb{R} \).

**Proof:** Factor \( p(z) = (z - r_1) \ldots (z - r_k)(z - i \lambda_1) \ldots (z - i \lambda_m) \) with \( \lambda_i \in \mathbb{C} \)

\( \lambda_1, \ldots, \lambda_m \) real \( \in \mathbb{R} \)

\( \lambda_{m+1}, \ldots, \lambda_k \in \mathbb{C} \setminus \mathbb{R} \)

Note that any root \( r = a + ib \in \mathbb{C} \) of \( p(z) \)

has conjugate \( \overline{r} = a - ib \in \mathbb{C} \) another root of \( p(z) \)

since \( 0 = p(\overline{r}) = \overline{r}^k + \sum_{j=0}^{k-1} a_j \overline{r}^j \)

\( 0 = \overline{0} = p(r) = r^k + \sum_{j=0}^{k-1} a_j r^j = \overline{r}^k + \sum_{j=0}^{k-1} a_j \overline{r}^j \)

Thus the roots \( r_1, \ldots, r_k \in \mathbb{C} \setminus \mathbb{R} \) must come in conjugate pairs \( a + ib, a - ib \) with \( b \neq 0 \)

and \( (z - (a + ib))(z - (a - ib)) \)

\( = z^2 - ((a + ib) + (a - ib))z + (a + ib)(a - ib) \)

\( = z^2 - 2az + (a^2 + b^2) \)

irreducible over \( \mathbb{R} \) when \( b \neq 0 \)

since \( \Delta = (2a)^2 - 4(a^2 + b^2) = -4b^2 < 0 \)
§ 1.7 Multivariate derivatives

- answering the question: given \( f: \mathbb{R}^n \to \mathbb{R}^m \), is there a linear function \( \mathbb{R}^n \to \mathbb{R}^m \) that best approximates \( f \) near some point \( \mathbf{x} = \mathbf{a} \in \mathbb{R}^n \), and how to compute it?

We have a good idea already for \( n = m = 1 \), i.e. \( f: \mathbb{R}^1 \to \mathbb{R}^1 \)

**Example:** \( f(x) = x^3 \) near \( x = 2 \)

\[
\begin{align*}
&f(x) = x^3 \quad \text{near } x = 2 \\
&f(x) = 3x^2 \\
&f'(x) = 3x^2 \text{ exists for all } x \in \mathbb{R}^1 \\
&f(2) = f'(2) = 3 \cdot 2^2 = 12 \\
&g(x) = f(x) - f(a) \\
&\frac{g(x) - g(a)}{x - a} = 3x^2 \\
&\lim_{h \to 0} \frac{g(x) - g(2)}{x - 2} = 12 \\
&g(x) = 12x - 16 \\
&\text{not linear } \mathbb{R}^1 \to \mathbb{R}^1 \quad \text{(affine-linear)}
\end{align*}
\]

- slope \( m = f'(2) = 12 \)

But we could have considered just as well \( f(x) = f(x + a) - f(a) \)

\[
\begin{align*}
&f(x) = (x + 2)^3 - 8 \\
&g(x) = mx = 12x \\
&\text{Linear} \\
&\text{slope } m = f'(a) = 12 \\
&y = f(x) \\
&y = g(x)
\end{align*}
\]

**Non-examples:**

- \( f(x) = |x| \) near \( x = 0 \)

\[
y = |x|
\]

- \( f(x) = x^3 \) near \( x = 0 \)

\[
\text{but we'll deal with it as an implicit function } x^3 = y \text{ later in Chap 2.}
\]

In fact, here's another equivalent definition of differentiability at \( a \) for \( f: \mathbb{R}^n \to \mathbb{R}^m \), that generalizes better to \( \mathbb{R}^n \to \mathbb{R}^m \):

**Definition:** For \( U \) open in \( \mathbb{R}^n \) and \( f: U \to \mathbb{R}^m \), \( f \) is differentiable at some \( \mathbf{a} \in U \), with \( f'(a) = m \), if

\[
\lim_{h \to 0} \frac{f(a + mh) - f(a) - mh}{h} = 0
\]

(a linear function \( \mathbb{R}^1 \to \mathbb{R}^m \) as \( h \to 0 \)).
Why equivalent to the usual?

\[ \lim_{h \to 0} \frac{f(a+h) - f(a) - mh}{h} = 0 \quad (\text{and exists, in particular}) \]

\[ \uparrow \quad \text{since} \quad \lim_{h \to 0} \frac{1}{h}(mh) = m \quad \text{exists, can add it to both sides} \]

\[ \lim_{h \to 0} \frac{1}{h}(f(a+h) - f(a) - mh) + \lim_{h \to 0} \frac{1}{h}(mh) = m \]

\[ \uparrow \quad \text{limit laws} \]

\[ \lim_{h \to 0} \frac{1}{h}(f(a+h) - f(a)) = m \quad \text{usual definition} \]

Rather than doing something naïve (and wrong) for \( \overline{f} : \mathbb{R}^n \to \mathbb{R}^m \), like defining \( f(x) = \lim_{h \to 0} \frac{\overline{f}(x+h) - \overline{f}(x)}{|h|} \) (wrong even for \( n = m = 1 \) since \( |h| \) is always positive)

we ask for a linear function \( L : \mathbb{R}^n \to \mathbb{R}^m \) that plays the above role of \( mh \)...

**Definition (7.10, essentially)** For \( \overline{f} : \mathbb{R}^n \to \mathbb{R}^m \) open and \( a \in U \),

\( \mathbb{R}^n \cap \mathbb{R}^m \)

say that \( \overline{f} \) is differentiable at \( a \) with derivative \( L \) if some linear transformation \( L : \mathbb{R}^n \to \mathbb{R}^m \)

\[ \times \ x \mapsto L(x) \]

with

\[ \lim_{h \to 0} \frac{1}{|h|}(f(a+h) - f(a)) - L(h) = 0. \]

In this case we write \( D\overline{f}(a) = L \)

(and we'll see shortly how to compute the matrix \( [D\overline{f}(a)] = [L] \)

that represents \( L : \mathbb{R}^n \to \mathbb{R}^m \), via partial derivatives & Jacobian matrix.)