Combining spanning & lin. independence gives an important concept.

**Def.** Given a subspace $E \subset \mathbb{R}^n$, a basis for $E$ is a subset $(\{v_1, ..., v_k\} \subset E$ that spans $E$ and is lin. indep.

Equivalently, $(\{v_1, ..., v_k\})$ is a basis for $E \iff$ every $v \in E$ has a unique expression

$$v = c_1v_1 + ... + c_kv_k.$$ 

**Example:** for a matrix $A$, the solutions to $Ax = 0$ always form a subspace of $\mathbb{R}^n$, and we can use row-reduction to find a basis.

E.g. $A = \begin{bmatrix} 1 & -1 & -1 \end{bmatrix}$

$$E := \{x \in \mathbb{R}^3: Ax = 0 \} = \{ \text{solns to } \begin{bmatrix} 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \}$$

$$= \{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = c_2 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \}$$

$$\Rightarrow E \text{ has basis } \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} : v_1, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} : v_2 \quad (Why?)$$

2. Every basis for $\mathbb{R}^n$ has exactly $n$ elements (e.g. $\{e_1, ..., e_n\}$)

**Prop:** For a subspace $E \subset \mathbb{R}^n$, and $(\{v_1, ..., v_k\} \subset E)$ TFAE

1. $(\{v_i\}_{i=1}^k$ are a basis for $E$
2. they are a minimal spanning set for $E$, i.e. removing any $v_i$ no longer spans $E$
3. they are a maximal lin. indep. set in $E$, i.e. adding any $v \in E$ ruins their lin. independence.

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(a) \(\Rightarrow\) (b): A basis spans, and if removing \(v_i\) it still spans, then \(v_i \in \text{span}\{v_j\}_{j \neq i}\)

\[ v_i = \sum_{j \neq i} c_j v_j \]

\[ 1 \cdot v_i - \sum_{j \neq i} c_j v_j = 0 \] gives a nontrivial dependence among \(\{v_i\}\).

(a) \(\Rightarrow\) (c): A basis is lin. indep., and if you add any \(v \in \mathbb{F}\) to it, then the expression \(v = \sum_{j=1}^{\text{dim}} c_j v_j\)

leads to a dependence among \(\{v_j\}_{j=1, \ldots, k} \cup \{v\} \).

(b) \(\Rightarrow\) (a): A minimal spanning set \(\{v_i\}_{i=1, \ldots, k}\) is also lin. indep., else a dependence \(c_1 v_1 + \ldots + c_k v_k = 0\) with \(c_k \neq 0\) (WLOG) leads to a smaller spanning set \(\{v_{1, \ldots, k-1}\}\), as \(v_k = \left(\frac{c_1 v_1 + \ldots + c_{k-1} v_{k-1}}{c_k}\right) \in \text{span}\{v_{1, \ldots, k-1}\}\).

(c) \(\Rightarrow\) (a): A maximal indep. set \(\{v_i\}_{i=1, \ldots, k}\) is also spanning, since if \(v \in \mathbb{F}\) has \(v \notin \text{span}\{v_{1, \ldots, k}\}\), one cannot have a nontrivial dependence among \(v_{1, \ldots, k} v\) as it would need a nonzero coefficient on \(v\), show \(v \notin \text{span}\{v_{1, \ldots, k}\}\). \(\Box\)

**COR (PROP-DEFN 2.4.19)**
Every subspace \(E \subseteq \mathbb{C}^n\) has a basis, and any two such bases contain the same number of vectors, called the dimension \(\dim(E)\) of \(E\).

**EXAMPLE:** \(E = \text{solns to } \begin{bmatrix} 1 & -1 & -1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0\) has \(\dim(E) = 2\) since it had basis \(\{[1], [0]\} \).
proof: To find a basis for \( E \), either \( E = \{ 0 \} \) and the empty set is a basis (Why?), or one can start with some \( \bar{v}_i \in E - \{ 0 \} \) and either \( E = \text{span}(\bar{v}_i) \)

or \( \exists \text{ some } \bar{v}_2 \in E \setminus \text{span}(\bar{v}_i) \) (so \( (\bar{v}_i, \bar{v}_2) \) are l.m. indep. - Why?)

and either \( E = \text{span}(\bar{v}_i, \bar{v}_2) \)

or \( \exists \text{ some } \bar{v}_3 \in E \setminus \text{span}(\bar{v}_i, \bar{v}_2) \) (so \( (\bar{v}_i, \bar{v}_2, \bar{v}_3) \) are l.m. indep. - Why?)

etc.

This must stop with \( E = \text{span}(\bar{v}_i, \bar{v}_2, \ldots, \bar{v}_k) \) for some \( k \leq n \) (since \( E \subset \mathbb{R}^n \))

and since \( \{ \bar{v}_i, \bar{v}_2, \ldots, \bar{v}_k \} \) are l.m. indep., they're a basis for \( E \).

Given 2 bases \((\bar{v}_1, \ldots, \bar{v}_m)\) for \( E \)

and \((\bar{w}_1, \ldots, \bar{w}_p)\)

uniquely express \( \bar{w}_j = \sum_{i=1}^{m} a_{ij} \bar{v}_i \) \( A = (a_{ij}) \)

and \( \bar{v}_i = \sum_{k=1}^{p} b_{ki} \bar{w}_k \) \( B = (b_{ki}) \)

Then one has

\[
\bar{w}_j = \sum_{i=1}^{m} a_{ij} \bar{v}_i = \sum_{i=1}^{m} a_{ij} \sum_{k=1}^{p} b_{ki} \bar{w}_k = \sum_{k=1}^{p} \left( \sum_{i=1}^{m} b_{ki} a_{ij} \right) \bar{w}_k
\]

up

coefficients on \( w_j \) must be identical, since \( \{ \bar{w}_j \} \) are l.m. indep.

\[
(BA)_{kj} = \begin{cases} 1 & \text{if } j = k \\ 0 & \text{otherwise} \end{cases}
\]

i.e. \( BA = I_p \)

Similarly using \( \bar{v}_i = \sum_{k=1}^{p} b_{ki} \sum_{l=1}^{m} a_{l} \bar{v}_l \), conclude \( AB = I_m \). Hence \( A = B \).
Some choices of bases for a subspace are more convenient than others...

**DEFINITION (2.4.16)**

A set of vectors \( \{ \vec{v}_1, ..., \vec{v}_k \} \) is **orthogonal** if \( \vec{v}_i \cdot \vec{v}_j = 0 \ \forall \ i \neq j \)

and it is **orthonormal** if additionally \( |\vec{v}_i| = 1 \ \forall i = 1, ..., k \)

i.e. \( \vec{v}_i \cdot \vec{v}_i = 1 \)

\( |\vec{v}_i|^2 \)

**EXAMPLES:**

1. \( \{ \vec{e}_1, ..., \vec{e}_n \} \) are orthonormal in \( \mathbb{R}^n \) since \( \vec{e}_i \cdot \vec{e}_j = 0 \ \forall i \neq j \)

\[ |\vec{e}_i| = 1 \ \forall i \]

2. \( \{ \vec{v}_1 = [1], \vec{v}_2 = [-1] \} \) are orthogonal, but not orthonormal,

however \( \{ \vec{u}_1 = \frac{1}{\sqrt{2}} [1], \vec{u}_2 = \frac{1}{\sqrt{2}} [1] \} \)

are orthonormal

**PROPOSITION (2.4.17)**

(i) Orthogonal sets of vectors \( \{ \vec{v}_1, ..., \vec{v}_k \} \) are always linearly independent, hence a basis for their span.

(ii) Orthonormal sets of vectors \( \{ \vec{v}_1, ..., \vec{v}_k \} \) have the convenient property that any \( \vec{v} \in \text{span}(\vec{v}_1, ..., \vec{v}_k) \)

has this expansion \( \vec{v} = \sum_{i=1}^{k} c_i \vec{v}_i \) given by the easily computed coefficients \( c_i = \vec{v} \cdot \vec{v}_i, \forall i = 1, ..., k \)

**proof:** For (i), if \( \vec{v}_1, ..., \vec{v}_k \) are orthogonal and one has a dependence \( c_1 \vec{v}_1 + ... + c_k \vec{v}_k = \vec{0} \), then for each \( j = 1, ..., k \) take dot product with \( \vec{v}_j \) to get

\[ c_j \vec{v}_j \cdot \vec{v}_j + ... + c_k \vec{v}_k \cdot \vec{v}_j = \vec{0} \cdot \vec{v}_j = 0 \]

\[ \Rightarrow c_j |\vec{v}_j|^2 = 0 \quad \text{since } \vec{v}_j \neq \vec{0} \]

\[ \Rightarrow c_j = 0. \]
For (c), if \( \mathbf{v}_i \), \( i = 1, \ldots, k \) are orthonormal,

given \( \mathbf{v} = c_1 \mathbf{v}_1 + \ldots + c_k \mathbf{v}_k \)

again for \( j = 1, \ldots, k \) take dot product with \( \mathbf{v}_j \) to get

\[
\mathbf{v} \cdot \mathbf{v}_j = c_1 \mathbf{v}_j \cdot \mathbf{v}_j + \ldots + c_j \mathbf{v}_j \cdot \mathbf{v}_j + \ldots + c_k \mathbf{v}_k \cdot \mathbf{v}_j
\]

\[
\Rightarrow \mathbf{v} \cdot \mathbf{v}_j = c_j
\]

**Example:** \( \left\{ \mathbf{u}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \mathbf{u}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\} \) is an orthonormal basis for \( \mathbb{R}^2 \)

and hence \( \mathbf{v} = \begin{bmatrix} 7 \\ 100 \end{bmatrix} = c_1 \mathbf{u}_1 + c_2 \mathbf{u}_2 \)

where \( c_1 = \mathbf{v} \cdot \mathbf{u}_1 = \begin{bmatrix} 7 \\ 100 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1/\sqrt{2} \end{bmatrix} \)

\[
= \frac{107}{\sqrt{2}}
\]

\( c_2 = \mathbf{v} \cdot \mathbf{u}_2 = \begin{bmatrix} 7 \\ 100 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1/\sqrt{2} \end{bmatrix} \)

\[
= -\frac{93}{\sqrt{2}}
\]

**Non-Example:** \( \left\{ \text{solutions to } \begin{bmatrix} 1 & -1 & -1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix} \text{ in } \mathbb{R}^3 \right\} \)

had basis \( \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\} \), but it is neither orthogonal nor orthonormal.