Remarks on spectral thm.

1. Restated: \( A = A^T \) real symmetric.
   \[ \iff A = P \Delta P \text{ for } P \text{ orthogonal (} P = P^T \), and } \Delta \text{ real diagonal } \]
   \[ \iff \text{proven} \]
   \[ \iff \text{note} \]
   \[ (P^T \Delta P)^T = P^T \Delta (P^T)^T = P^T \Delta P \]

2. Similar proof strategy via induction on \( n \) would have proven:
   \[ \text{THM} \]
   A \( n \times n \) C-matrix is normal, meaning \( A(A^T) = (A^T)A \),
   \[ \iff A = U \Delta U \text{ for } U \text{ unitary (} U^* = \bar{U}^T \), and } \Delta \text{ complex diagonal } \]
   \[ \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} \lambda_i \in \mathbb{C} \]

3. (Further)
   Similar strategy would prove:
   \[ \text{THM} \]
   every \( A \times n \) C-matrix can be triangularized by an invertible matrix \( P \) over \( \mathbb{C} \), i.e.
   \[ P^T \Delta P = \begin{bmatrix} \lambda_1 & * & \cdots & * \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix} \]
   … but in fact it’s worth looking up a statement and/or quick proof of the Jordan canonical form for \( A \), which is a much more precise triangular form for \( A \).

4. Spectral thm is closely related to singular value de-composition (\( \text{SVD} \))
   (important for principal component analysis - \( \text{PCA} \))
   of a rectangular real \( X = P \Sigma Q \)
   \[ \text{by applying spectral thm to } P^T P = \tilde{Q} \Sigma \tilde{Q} = Q \Sigma Q \]
   or \( A_2 = XX^T = P \Sigma^2 P \).
Recall how this root-finding method works in one variable:

Linear approximation to \( y = f(x) \) at \( x = a_0 \) has equation

\[
y - f(a_0) = f'(a_0)(x - a_0)
\]

so solve for \( y = 0 \) here to get the \( x \)-value \( a_1 \) for the approximate root:

\[
x - a_0 = -\frac{f(a_0)}{f'(a_0)}
\]

\[
x = -\frac{f'(a_0)^{-1}f(a_0) + a_0}{f'(a_0)}
\]

i.e. let \( a_1 \) be this.

Now repeat to get \( a_2, a_3, \ldots \).

Note that we needed \( f'(a_0) \neq 0 \) so that we could divide by it.

The multivariate version is analogous.

**DEF’N 2.8.1 (multivariate Newton’s method)**

When looking for a solution to \( n \) equations in \( n \) unknowns \( \mathbf{x} = [x_1, \ldots, x_n]^T \):

\[
\begin{align*}
\frac{f_1(x)}{f_2(x)} &= 0 \\
\vdots & \quad \vdots \\
\frac{f_n(x)}{f_m(x)} &= 0
\end{align*}
\]

if we regard \( \mathbf{f}(\mathbf{x}) \) as a map \( f: U \rightarrow \mathbb{R}^n \), and it is differentiable at some \( \mathbf{a}_0 \in U \) with \( \mathbf{D}f(\mathbf{a}_0) \) invertible, we can try to approximate a solution by instead solving the linear system \( \mathbf{y} - \mathbf{f}(\mathbf{a}_0) = \mathbf{D}f(\mathbf{a}_0)(\mathbf{x} - \mathbf{a}_0) \) for \( \mathbf{y} = \mathbf{0} \), i.e.

\[
\begin{align*}
\text{find } \mathbf{x} \text{ such that } &\mathbf{D}f(\mathbf{a}_0)(\mathbf{x} - \mathbf{a}_0) = -\mathbf{f}(\mathbf{a}_0) \\
(\text{or equivalently } &\mathbf{a}_1 = -\mathbf{D}f(\mathbf{a}_0)^{-1}\mathbf{f}(\mathbf{a}_0) + \mathbf{a}_0)
\end{align*}
\]
EXAMPLE: Where do the circle $x^2 + y^2 = 10$ and hyperbola $xy = 1$ intersect?

We could solve this directly via $y = \frac{1}{x}$ and substituting $x^2 + \left(\frac{1}{x}\right)^2 = 10$

$$x^4 + 1 = 10x^2$$

$$x^4 - 10x^2 + 1 = 0$$

$$x^2 = \frac{10 \pm \sqrt{100 - 4}}{2}$$

$$x = \pm \sqrt{5 \pm \sqrt{21}}$$

We get $(8.14626, 0.317837)$ and $(0.317837, 3.14626)$

Now let's try Newton's method for solving

$$\begin{cases} f_1(x, y) = x^2 + y^2 - 10 \\ f_2(x, y) = xy - 1 \end{cases}$$

so we have

$$\mathbb{R}^2 \xrightarrow{\bar{f}} \mathbb{R}^2$$

$$\bar{x} = (x, y) \ xrightarrow{\bar{f}(\bar{x}) = \begin{pmatrix} f_1(\bar{x}) \\ f_2(\bar{x}) \end{pmatrix} = \begin{pmatrix} x^2 + y^2 - 10 \\ xy - 1 \end{pmatrix}}$$

with derivative

$$\begin{pmatrix} x \\ y \end{pmatrix} \ xrightarrow{\text{Jacobian} \ \bar{f}(\bar{x})} \begin{pmatrix} 2x & 2y \\ y & x \end{pmatrix}$$

As an initial guess, if we try $\bar{a}_0 = (1, 1)$, we have $\bar{f}(\bar{a}_0) = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ not invertible, can't use Newton!

If we try $\bar{a}_0 = (2)$, we have $\bar{f}(\bar{a}_0) = \begin{pmatrix} 4 & 2 \\ 1 & 2 \end{pmatrix}$ invertible,

and next guess $\bar{a}_1 = -\bar{f}(\bar{a}_0)^{-1} \bar{f}(\bar{a}_0) + \bar{a}_0 = -\begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix}^{-1} \begin{pmatrix} 4 \\ 2 \end{pmatrix} + \bar{a}_0 = \begin{pmatrix} \frac{3}{2} \\ 1 \end{pmatrix}$

then $\bar{a}_2 \approx \begin{pmatrix} 3.3 \\ 0.16 \end{pmatrix}$, $\bar{a}_3 \approx \begin{pmatrix} 3.1547 \\ 0.317837 \end{pmatrix}$, $\bar{a}_4 \approx \begin{pmatrix} 3.14629 \\ 0.317837 \end{pmatrix}$, $\bar{a}_5 \approx \begin{pmatrix} 3.14626 \\ 0.317837 \end{pmatrix}$

(same as above!)