DEFN 2.6.10-2.6.14: As before, we define

- Lin. independence of \( \{ \bar{v}_i : i = 1, \ldots, k \} \) in \( V \), i.e., \( \sum_{i=1}^k c_i \bar{v}_i = 0 \Rightarrow c_1 = \ldots = c_k = 0 \)
- \( \text{span}(\bar{v}_1, \ldots, \bar{v}_k) = \{ c_1 \bar{v}_1 + \ldots + c_k \bar{v}_k : c_i \in \mathbb{R} \} \)
- \( \{ \bar{v}_i : i = 1, \ldots, k \} \) are a basis for \( V \) if they are lin. indep. & \( \text{span}(\bar{v}_i : i = 1, \ldots, k) = V \)

or equivalently, every \( \bar{v} \in V \) can be written uniquely as \( \bar{v} = \sum_{i=1}^k c_i \bar{v}_i \)

(in which case, you call \( [\begin{array}{c} x_1 \\ \vdots \\ x_m \end{array}] \) the coordinates of \( \bar{v} \) with respect to the ordered basis \( (\bar{v}_1, \ldots, \bar{v}_m) \))

Once you have picked (ordered) bases \( (\bar{v}_1, \ldots, \bar{v}_m) \) for \( V \)
\( (\bar{w}_1, \ldots, \bar{w}_n) \) for \( W \)

one can express any linear transformation uniquely in these bases via an \( m \times n \) matrix \( A = \left[ \begin{array}{c} a_{1j} \\ \vdots \\ a_{mj} \end{array} \right] \)

where \( T(\bar{v}_j) = \sum_{i=1}^m a_{ij} \bar{w}_i \)

that is, \( A = \left[ \begin{array}{cccc} T(\bar{v}_1) & T(\bar{v}_2) & \cdots & T(\bar{v}_m) \end{array} \right] \)

where \( T(\bar{v}_j) \) is expressed in coordinates with respect to \( (\bar{w}_1, \ldots, \bar{w}_n) \)

As before, composition of linear maps com. to multiplying matrices.

EXAMPLES:

1. \( P_d = \{ a_0 + a_1 x + \ldots + a_d x^d \} \) has basis \( (1, x, x^2, \ldots, x^d) \)

\( P_{d-1} \) has basis \( (1, x, x^2, \ldots, x^{d-1}) \)

The linear transformation \( P_d \xrightarrow{\frac{d}{dx}} P_{d-1} \) expressed with respect to these choices of bases has matrix \( A \)

\[ A = \frac{d}{dx} \]

eg. \( d = 3 \)
2. The integration map \( P_{d-1} \overset{\int^x}{\longrightarrow} P_d \)

\[
p(x) \quad \mapsto \quad \int_0^x p(t) \, dt
\]

is also linear (Why?)

and with respect to the same chosen of bases has matrix

\[
\begin{pmatrix}
1 & x & x^2 & \cdots & x^{d-1} \\
1 & 0 & 0 & & 0 \\
x & 1 & 0 & & 0 \\
x^2 & 0 & \frac{1}{2} & 0 & 0 \\
x^3 & 0 & 0 & 4 & 0 \\
\vdots & & & \ddots & \vdots \\
x^{d-1} & 0 & 0 & 0 & \cdots & 0 \\
x^d & 0 & 0 & 0 & \cdots & 1 \\
\end{pmatrix}
\]

\[
\text{eg. } B = \begin{pmatrix}
1 & x & x^2 \\
1 & 1 & 0 \\
x & 0 & \frac{1}{2} \\
x^2 & 0 & 0 \\
x^3 & 0 & 0 & \frac{1}{3}
\end{pmatrix}
\]

3. Since the composite map

\[
P_{d-1} \overset{\int^x}{\longrightarrow} P_d \overset{\frac{\partial}{\partial x}}{\longrightarrow} P_{d-1}
\]

\[
p(x) \quad \mapsto \quad \int_0^x p(t) \, dt \quad \mapsto \quad \frac{\partial}{\partial x} \left( \int_0^x p(t) \, dt \right) = p(x)
\]

is the identity map \( 1_{P_d} \), one should have \( AB = I_d \)

\[
\text{(i.e. } \frac{\partial}{\partial x} \circ \int_0^x = 1_{P_d} \text{)}
\]

and this is what happens, e.g. for \( d=3 \)

\[
\begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 3
\end{pmatrix}
\begin{pmatrix}
0 & 0 & 0 \\
1 & 0 & 0 \\
0 & 1\frac{1}{2} & 0
\end{pmatrix}
= \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
= I_3
\]

\( A \) has \( B \) as a right-inverse; \( \frac{\partial}{\partial x} : P_d \rightarrow P_{d-1} \) is surjective (but not injective; who is \( \ker(\frac{\partial}{\partial x}) \)?)

\( B \) has \( A \) as a left-inverse; \( \int^x : P_{d-1} \rightarrow P_d \) is injective (but not surjective; who is not in \( \text{im}(\int^x) \)?)
Similarly one could do the composite in the other order:

\[ \begin{align*}
\mathcal{P}_d & \xrightarrow{\frac{d}{dx}} \mathcal{P}_{d-1} \xrightarrow{\left( \begin{bmatrix} 0 & x & x^2 & \ldots & x^{d-1} \end{bmatrix} \right)} \\
\mathcal{P}_d & \xrightarrow{\frac{d}{dx}} \mathcal{P}_{d-1} \xrightarrow{\left( \begin{bmatrix} 0 & x & x^2 & \ldots & x^{d-1} \end{bmatrix} \right)} 
\end{align*} \]

The composite \[ \mathcal{P}_d \xrightarrow{\frac{d}{dx}} \mathcal{P}_{d-1} \xrightarrow{\left( \begin{bmatrix} 0 & x & x^2 & \ldots & x^{d-1} \end{bmatrix} \right)} \]

has matrix (w.r.t. to the ordered basis \( \{1, x, x^2, \ldots, x^d\} \) for \( \mathcal{P}_d \))

\[ BA = \left( \begin{bmatrix} 0 & x & x^2 & \ldots & x^{d-1} \end{bmatrix} \right) \left( \begin{bmatrix} 1 & 0 & 0 & \ldots & 0 \\
0 & 1 & 0 & \ldots & 0 \\
0 & 0 & 1 & \ldots & 0 \\
0 & 0 & 0 & \ldots & 1 
\end{bmatrix} \right) = x \left( \begin{bmatrix} 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 
\end{bmatrix} \right) \]

\[ \implies \left( \int_0^x \frac{d}{dx} (x^m) \right)(x^n) = \begin{cases} x^{m+n} & \text{if } m > 0 \\
0 & \text{if } m = 0 
\end{cases} \]

---

**DEFINITION 2.65**

An **isomorphism** of vector spaces \( V, W \)

is a **bijective** linear transformation \( T: V \rightarrow W \)

(injective, surjective)

\( \ker(T) = \{0\} \), \( \operatorname{img}(T) = W \)

Picking an ordered basis \( \{v_1, \ldots, v_n\} \) for \( V \)

leads to the isomorphism \( \mathbb{R}^n \xrightarrow{\Phi_{v_1}} V \)

\[ \begin{bmatrix} x_1 \\
\vdots \\
x_n \end{bmatrix} \rightarrow x_1v_1 + \ldots + x_nv_n \]

which the book calls the "concrete to abstract" function.

\( \Phi^{-1}_{v_1}(\bar{w}) \) gives the coordinates \[ \begin{bmatrix} x_1 \\
\vdots \\
x_n \end{bmatrix} \] of \( \bar{w} \) with respect to the ordered basis \( \{v_1, \ldots, v_n\} \)

i.e. \( \bar{w} = x_1v_1 + \ldots + x_nv_n \)

What is interesting is how one changes the coordinates in \( V \) from one choice of an ordered basis \( \{v_1, \ldots, v_n\} \) to another one \( \{\bar{w}_1, \ldots, \bar{w}_n\} \)

First, let's check that we should have \( m=n \) here.
**Theorem 2.6.22:** If \( \{\tilde{v}_1, \ldots, \tilde{v}_n\}, \{\tilde{w}_1, \ldots, \tilde{w}_m\} \) are bases for \( V \), then \( n = m = \dim(V) \).

**Proof:** We'd like to say that the composite bijection
\[
\mathbb{R}^n \xrightarrow{\Phi_{i j}^{-1}} V \xrightarrow{\Phi_{j i}^{-1}} \mathbb{R}^m
\]

is a linear transformation \( \mathbb{R}^n \rightarrow \mathbb{R}^m \), which is bijective; hence \( n = m \).

However, first we should check that \( \Phi_{i j}^{-1} \) is linear,

by noting **Lemma:** If \( T : V \rightarrow W \) is a bijective linear map

then \( T^{-1} : W \rightarrow V \) is also linear.

**Proof:** (same as for Prop 1.3.14):

Want to check \( T^{-1}(a\tilde{v}_1 + b\tilde{v}_2) = aT^{-1}(\tilde{v}_1) + bT^{-1}(\tilde{v}_2) \),

but since \( T \) is bijective, this is true if and only if

\[
T^{-1}(a\tilde{v}_1 + b\tilde{v}_2) = T^{-1}(aT^{-1}(\tilde{v}_1) + bT^{-1}(\tilde{v}_2))
\]

Thus \( \Phi_{i j}^{-1} \) is linear,

so the composite \( \Phi_{i j}^{-1} \circ \Phi_{j i} : \mathbb{R}^n \rightarrow \mathbb{R}^m \) is linear, bijective,

forcing \( n = m \).

In fact, the matrix that represents
\[
\mathbb{R}^n \xrightarrow{\Phi_{i j}^{-1} \circ \Phi_{j i}} \mathbb{R}^n
\]

is the matrix that converts the coordinates w.r.t. \( \tilde{v}_j \) for a vector in \( V \)

to its coordinates w.r.t. \( \tilde{w}_j \):

\[
\begin{bmatrix}
\tilde{v}_1^T & \cdots & \tilde{v}_n^T
\end{bmatrix}
\begin{bmatrix}
\Phi_{i j}^{-1} \\
\Phi_{j i}
\end{bmatrix}
= \begin{bmatrix}
\begin{bmatrix}
\tilde{y}_1 \\
\vdots \\
\tilde{y}_m
\end{bmatrix}
\end{bmatrix}
\]

i.e., \( \Phi_{i j}^{-1} \) sends \( \tilde{v}_i \rightarrow \tilde{y}_i \).