Let's solve the very simple IVP

\[
\begin{align*}
\dot{x} &= \frac{dx}{dt} = ax, \quad (a \in \mathbb{R}) \\
\dot{x}(0) &= x_0.
\end{align*}
\]

By separation of variables, \( x(t) = x(0)e^{at} \). Note that this is the only function that satisfies the IVP.

Now let's try

\[
\begin{align*}
\dot{x_1} &= a_1 x_1 \\
\dot{x_2} &= a_2 x_2
\end{align*}
\]

which has the solution \( x_1(t) = x_1(0)e^{a_1t} \) and \( x_2(t) = x_2(0)e^{a_2t} \). This can be rewritten as

\[
\begin{bmatrix}
\dot{x_1} \\
\dot{x_2}
\end{bmatrix} =
\begin{bmatrix}
a_1 & 0 \\
0 & a_2
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix}
\]

and the solution can be written as

\[
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix} =
\begin{bmatrix}
e^{a_1t} & 0 \\
0 & e^{a_2t}
\end{bmatrix}
\begin{bmatrix}
x_1(0) \\
x_2(0)
\end{bmatrix}.
\]

In general, the system could be coupled, and we can express the (linear) system as \( \dot{x} = Ax \), where \( x \in \mathbb{R}^n \)

and \( A \) is an \( n \times n \) matrix.

**Fact:** Suppose that \( A \) has real eigenvalues and that they are distinct. Then the IVP \( \{ \dot{x} = Ax \} \) has the unique solution \( x(t) = e^{At}x_0 \).

**Example:**

\[
\begin{bmatrix}
\dot{x_1} \\
\dot{x_2}
\end{bmatrix} =
\begin{bmatrix}
-1 & -3 \\
0 & 2
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix}.
\]

After diagonalizing the matrix, we obtain

\[
x(t) =
\begin{bmatrix}
1 & 1 \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
e^{-t} & 0 \\
e^{-2t} & e^{-t}
\end{bmatrix}
\begin{bmatrix}
x_1(0) \\
x_2(0)
\end{bmatrix}.
\]

**Note:** This agrees with Fact because \( e^{At} = P^{-1}e^{A_{diag}t}P \) when \( P^{-1}AP = A_{diag} \) diagonal.