Implicit Function Thm

Suppose we are looking at a solution set in $\mathbb{R}^{n+m}$ of $n$ equations in the $n+m$ unknowns.

e.g. in $\mathbb{R}^{2}$, $F(x) = 0$

express $x$ in terms of $y$ here?

We might expect at most points on the solution set, we can pick a neighborhood and $m$ ("nonpivotal") variables $y_1, \ldots, y_m$ locally that let us express the $n$ ("pivotal") variables left $x_1, \ldots, x_n$ as $x = g(y)$.

Parameterize

e.g. in $\mathbb{R}^2$, on solution set $F(y) = x^2 + y^2 - 4$

near $\bar{c} = (a, b)$

one can either express $x = +\sqrt{4-y^2}$

or $-\sqrt{4-y^2}$

and also $y = +\sqrt{4-x^2}$

or $-\sqrt{4-x^2}$

But at $x = (\pm 2, 0)$, one can only express $x = -\sqrt{4-y^2}$ in a neighborhood around it:

(Can't decide $y = +\sqrt{4-x^2}$ or $-\sqrt{4-x^2}$ in any neighborhood around it.)
THM (Implicit Function Thm) Given $U \subseteq \mathbb{R}^n$ open in $C'(U)$

and a point $\bar{c} \in U$ where $DF(\bar{c}): \mathbb{R}^{n+m} \to \mathbb{R}^m$ is onto (= Full-rank, = surjective)

if one relabels the variables in $\mathbb{R}^{n+m}$ as $\begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix} = \begin{pmatrix} x' \\ y' \end{pmatrix}$ so that

$DF(\bar{c}): \begin{bmatrix} \bar{x}' \\ \bar{y}' \end{bmatrix}$ has $\bar{x}'$, $\bar{x}$ as pivot columns, then write $\bar{c} = \begin{pmatrix} a \\ b \end{pmatrix}$

with $\bar{a} \in \mathbb{R}^n$, $\bar{b} \in \mathbb{R}^m$, there are neighborhoods $A \subseteq \mathbb{R}^n$, $B \subseteq \mathbb{R}^m$ with $\bar{a} \in A$, $\bar{b} \in B$ such that $f(\bar{g}(\bar{y})) = \bar{c}$ and $\bar{g}(\bar{y}) = \bar{a}$ open neighborhoods of $\bar{a}$ in $\mathbb{R}^n$ and $\bar{b}$ in $\mathbb{R}^m$ thereby making $\bar{g}$ differentiable such that $F(\bar{g}(\bar{y})) = \bar{c} \quad \forall \bar{y} \in B$.

\[ i.e. \quad \bar{x} = \bar{g}(\bar{y}) \text{ expresses } \bar{x} \text{ in terms of } \bar{y} \text{ around } \bar{c} = \begin{pmatrix} a \\ b \end{pmatrix} \text{ on } F(\bar{g}) = \bar{c}. \]

(proof in a while...)

EXAMPLES:

1. $\mathbb{R}^2 \rightarrow \mathbb{R}^n$ has $JF(\bar{y}) = \begin{bmatrix} 2x \\ 2y \end{bmatrix}$

$F(\bar{y}) = x^2+y-1$

so $JF(\bar{b}) = \begin{bmatrix} 2x \\ 2y \end{bmatrix}$ has either $x$ or $y$ as pivot variables

but at $\begin{pmatrix} \bar{g} \\ \bar{b} \end{pmatrix} = \begin{pmatrix} \pm 2 \\ 0 \end{pmatrix}$, $JF(\bar{g}) = \begin{bmatrix} \pm 4 \\ 0 \end{bmatrix}$

only can have $x$ as a pivot variable,

so one can only deduce from Imp Fn Thm

that $F$取得極值 \begin{pmatrix} \pm 2 \\ 0 \end{pmatrix} where $x(\pm \sqrt{4-1})$

only exists.

2. Worse things can happen, e.g.

$\mathbb{R}^2 \rightarrow \mathbb{R}^1$

$F(\bar{x}) = y^3 - x^2(x+1)$ defines a nodal cubic curve via $F(\bar{x}) = 0$.

\[ y^3 - x^2(x+1) = 0 \]

\begin{tikzpicture}
  \draw [->] (-3,0) -- (3,0) node [right] {$x$};
  \draw [->] (0,-3) -- (0,3) node [above] {$y$};
  \draw [domain=-2:2, samples=100] plot (\x, {\x^3 - \x^2*(\x+1)});
  \filldraw (-1,0) circle (2pt);
  \filldraw (0,-1) circle (2pt);
\end{tikzpicture}
Examining \( JF(x, y) = \left[ \begin{array}{c} \frac{\alpha}{-3x^2 + 2x} \\ \frac{\beta y}{-3x^2 + 2x} \end{array} \right] \), one sees that for most (b) on the curve, both variables \( x, y \) are pivotal with \( a \neq 0 \) and one can write \( x = g(y) \) or \( y = g(x) \).

However, when \( x = -\frac{2}{3} \), \( JF(x) = \left[ \begin{array}{c} \frac{\alpha}{3} \\ \frac{\beta}{3} \end{array} \right] \) and one can only write \( y = g(x) \).

When \( y = 0 \), \( x = -1 \), \( JF(-1) = \left[ \begin{array}{c} 0 \\ 0 \end{array} \right] \), and one can only write \( x = g(y) \).

When \( x = 0 \), \( y = 0 \), \( JF(0) = \left[ \begin{array}{c} 0 \\ 0 \end{array} \right] \), so we don't get anything from Imp. Fn. Thm.

**Example 2.10.6**

The 5-variable system \[
\begin{cases}
x^2 - y = a \\
y^2 - z = b \\
z^2 - x = c
\end{cases}
\]
has \( \tilde{c} = \left( \begin{array}{c} b \\ x \\ y \\ z \\ 0 \end{array} \right) \) as a solution. Near \( \tilde{c} \), can we express \( \left( \begin{array}{c} z \\ y \\ x \end{array} \right) \) in terms of \( \left( \begin{array}{c} 0 \\ 0 \\ 0 \end{array} \right) \) on this solution?

Here \( F: \mathbb{R}^5 \rightarrow \mathbb{R}^3 \)

\( F(x, z, y, a, b) = \left( \begin{array}{c} x^2 - y - a \\ y^2 - z - b \\ z^2 - x - c \end{array} \right) \) has \( JF = \left[ \begin{array}{ccc} 2x & -1 & 0 & -1 & 0 \\ 0 & 2y & -1 & 0 & -1 \\ -1 & 0 & 2z & 0 & 0 \end{array} \right] \).

\( JF(c) = \left[ \begin{array}{ccc} 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{array} \right] \), so **YES**, by Imp. Fn. Thm.

**YES: pivot columns!**

(One could also express \( x \) in terms of \( z \), for example.)