Math 3592H Honors Math I

Instructions: Quiz 2, Thursday Oct. 20, 2016
15 minutes, closed book and notes, no electronic devices.
There are three problems, worth a total of 20 points.

1. (8 points total; 4 points each part)

Consider the function \( f : \mathbb{R}^3 \to \mathbb{R}^2 \) defined by \( f \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} xy \\ z^2 \end{pmatrix} \).

(a) Write down the matrix representing \( Df(a) \) at \( a = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \).

Since \( \bar{f} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \) has both \( f_1, f_2 \) polynomial, it is differentiable everywhere on \( \mathbb{R}^3 \), with

\[
\left[D\bar{f}(1)\right] = \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z} \end{bmatrix} = \begin{bmatrix} y & 0 & 0 \\ 0 & 0 & 2z \end{bmatrix}
\]

(b) Compute the directional derivative of \( f \) at \( a = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \) in the direction of the unit vector \( \mathbf{v} = \begin{pmatrix} 3/5 \\ 4/5 \\ 0 \end{pmatrix} \).

Since \( \bar{f} \) is differentiable at \( (1) \), this directional derivative is

\[
\left[D\bar{f}(1)\right] \mathbf{v} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 3/5 \\ 4/5 \\ 0 \end{bmatrix} = \begin{bmatrix} 7/5 \\ 0 \end{bmatrix}
\]
2. (8 points total; 4 points each part)
Assume \( f : \mathcal{U} \rightarrow \mathbb{R}^{100} \) is defined on an open subset \( \mathcal{U} \) of \( \mathbb{R}^3 \), and differentiable at some point \( a \in \mathcal{U} \).

(a) What are the dimensions of the Jacobian matrix \( Jf(a) \)?
\[
Jf(a) \text{ represents a linear transformation } \mathbb{R}^3 \rightarrow \mathbb{R}^{100}, \text{ so it is } 100 \times 3 \text{ columns.}
\]

(b) On what subset of points \( x \in \mathbb{R}^3 \) is the derivative \( Df(a)(x) \) defined?
\[
Df(a) : \mathbb{R}^3 \rightarrow \mathbb{R}^{100}, \text{ i.e. it is defined on all of } \mathbb{R}^3.
\]

3. (4 points total) Prove or disprove:
The function \( f : \text{Mat}(n,n) \rightarrow \text{Mat}(n,n) \) sending \( X \in \text{Mat}(n,n) \) to
\[
f(X) = I + 6X - 5X^2
\]
is differentiable on all of \( \text{Mat}(n,n) \), and at \( X = A \), its derivative 
\( Df(A) : \text{Mat}(n,n) \rightarrow \text{Mat}(n,n) \) is
\[
Df(A)(H) = 6H - 5(AH + HA).
\]

True, since, e.g.
\[
\lim_{H \rightarrow 0} \frac{f(A + H) - f(A) - (6H - 5(AH + HA))}{|H|} = 0
\]
\[
\lim_{H \rightarrow 0} \frac{I + 6(A + H) - 5(A + H)^2 - (I + 6A - 5A^2) - (6H - 5(AH + HA))}{|H|} = 0
\]
\[
\lim_{H \rightarrow 0} \frac{I + 6A + 6H - 5A^2 - 5AH - HA - 5A^2 - 5AH + 5HA}{|H|} = 0
\]
\[
-5 \lim_{H \rightarrow 0} \frac{H^2}{|H|} = 0
\]
Since \( |\frac{H^2}{|H|}| \leq |\frac{|H|^2}{|H|} = |H| \rightarrow 0 \) as \( H \rightarrow 0 \)

(or you could appeal to \( g(x) = x^2 \) having derivative \( Dg(A)(H) = AH + HA \)
and using limit laws for sums, scalings, etc.)