1. (30 points; 10 points each part)

Let \( A \) be a \( 3 \times 5 \) matrix.

(i) Prove or disprove: there are no vectors \( \mathbf{b} \) in \( \mathbb{R}^3 \) for which \( A\mathbf{x} = \mathbf{b} \) has exactly one solution \( \mathbf{x} \) in \( \mathbb{R}^5 \).

(ii) Now assume \( A \) can be row-reduced to \( \tilde{A} = \begin{bmatrix} 0 & 1 & 2 & -3 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \).

Write down a basis for the subspace \( V = \{ \mathbf{x} \in \mathbb{R}^5 : A\mathbf{x} = \mathbf{0} \} \).
(iii) Write down a matrix $E$ having the following property:

If $A = \begin{bmatrix} r_1^T \\ r_2^T \\ r_3^T \end{bmatrix}$ with $r_i$ in $\mathbb{R}^5$, then $EA = \begin{bmatrix} r_1^T \\ r_2^T \\ r_3^T - 6r_1^T \end{bmatrix}$

2. (20 points total) Prove or disprove: If $\overline{f}, \overline{g} : \mathbb{R}^4 \to \mathbb{R}^4$ are both differentiable everywhere, and $(\overline{f} \circ \overline{g}) \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} x_4 \\ x_3 \\ x_2 \\ x_1 \end{pmatrix}$ for all $\mathbf{x}$ in $\mathbb{R}^4$,

then the Jacobian matrix $[J\overline{f}(\mathbf{a})]$ is invertible for every$^1 \mathbf{a}$ in $\text{img}(\overline{g})$.

$^1$The exam had “for every $\mathbf{a}$ in $\mathbb{R}^4$”, which is not the assumption I intended!
3. (20 points total; 10 points each part) \( A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 3 & 4 & \alpha \end{bmatrix} \).

(i) Assuming that \( A\mathbf{x} = \mathbf{0} \) has infinitely many solutions, what is \( \alpha \)?

(ii) Assuming that \( \alpha \) is chosen as in the answer to part (i), write down at least one explicit \( \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \) in \( \mathbb{R}^3 \) so that \( A\mathbf{x} = \mathbf{b} \) has no solutions.
4. (30 points total; 10 points each part) Prove or disprove:

(a) If $\vec{v}_1, \vec{v}_2$ are nonzero, nonparallel vectors in $\mathbb{R}^3$, then $\{\vec{v}_1, \vec{v}_2, \vec{v}_1 \times \vec{v}_2\}$ are linearly independent.

(b) For any angle $\theta$, the vectors

\[ \vec{v}_1 = \begin{bmatrix} -\cos(6\theta) \\ -\sin(6\theta) \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} \sin(6\theta) \\ -\cos(6\theta) \end{bmatrix} \]

are orthonormal in $\mathbb{R}^2$.

(b) For any angle $\theta$, the vectors

\[ \vec{v}_1 = \begin{bmatrix} \cos(\theta) \\ \sin(\theta) \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} \cos(2\theta) \\ \sin(2\theta) \end{bmatrix} \]

are orthonormal in $\mathbb{R}^2$. 