Math 3592H Honors Math I
Midterm exam 1, Thursday October 6, 2016

Instructions:
50 minutes, closed book and notes, no electronic devices.
There are four problems, worth a total of 100 points.

1. (48 points total; 8 points each part)
For these vectors in \( \mathbb{R}^3 \),
\[
\vec{u} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \vec{v} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \vec{w} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix},
\]
compute the following via dot and cross products. Your answers are allowed to contain unevaluated inverse trigonometric functions.

(i) The length of \( \vec{u} \).

(ii) The angle between \( \vec{u}, \vec{v} \).

(iii) The length of the projection of \( \vec{v} \) orthogonally (perpendicularly) onto the line spanned by \( \vec{u} \).
(iv) The area of the parallelogram in $\mathbb{R}^3$ spanned by $\mathbf{u}$ and $\mathbf{v}$, that is, having vertices \{0, $\mathbf{u}$, $\mathbf{v}$, $\mathbf{u} + \mathbf{v}$\}.

(v) Some vector in $\mathbb{R}^3$ orthogonal (perpendicular) to both $\mathbf{u}$ and $\mathbf{v}$.

(vi) The volume of the parallelepiped (slanted box) in $\mathbb{R}^3$ spanned by $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$, that is, having vertices \{0, $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$, $\mathbf{u} + \mathbf{v}$, $\mathbf{u} + \mathbf{w}$, $\mathbf{v} + \mathbf{w}$, $\mathbf{u} + \mathbf{v} + \mathbf{w}$\}.
2. (21 points total; 7 points each part)
Assuming that \( f(x) = \sin(x) \) is continuous, and \( \lim_{x \to 0} \frac{\sin(x)}{x} = 1 \), compute with proof and/or explanations the values of the following limits of functions \( g : \mathbb{R}^3 \to \mathbb{R} \) as \( \vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \) approaches \( \vec{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \).

(i) \[ \lim_{x \to 0} 3x^2 + 5y + z \]

(ii) \[ \lim_{x \to 0} \sin(3x^2 + 5y + z) \]

(iii) \[ \lim_{x \to 0} \frac{\sin(3x^2 + 5y + z)}{3x^2 + 5y + z} \]
3. (15 points total) Consider the matrix \( A = \begin{bmatrix} 0 & a & b \\ 0 & 0 & a \\ 0 & 0 & 0 \end{bmatrix} \).

(i) (5 points) Compute \( A^2 \) and \( A^3 \).

(ii) (10 points) Compute \( e^A = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots \).

4. (16 points total; 8 points each part) Prove or disprove:

(a) An arbitrary union of (possibly infinitely many) closed sets is closed.

(b) If \( \lim_{k \to \infty} a_k = L \) in \( \mathbb{R} \), and \( a_k \leq M \) for all \( k \), then \( L \leq M \).