1 Math 8401: Assignment 1

1.1 Part I. Scaling and non-dimensionalization

1. In a classical work modeling the outbreak of the spruce bud worm in Canada’s balsam fir forests, researchers proposed that the bud worm population \( n = n(t) \) was governed by the law

\[
\frac{dn}{dt} = rn(1 - \frac{n}{K}) - P(n),
\]

where \( r \) and \( K \) are the growth rate and carrying capacity, respectively, and \( P(n) \) is a bird predation term given by

\[
P(n) = \frac{bn^2}{a^2 + n^2},
\]

where \( a \) and \( b \) are positive constants.

- Determine the dimensions of the constants \( a \) and \( b \).
- Graph the predation rate \( P(n) \) for \( a = 1, 5, 10 \) and make a qualitative statement about the effect that the parameter \( a \) has on the model.
- Select dimensionless variables \( N = \frac{n}{a} \) and \( \tau = \frac{b}{a}t \) and reduce the differential equation to dimensionless form.
- Find the equilibrium solutions of the dimensionless equation. (There are multiple solutions for selected parameter values.)

2. The length \( L \) of an organism depends upon time \( t \), its density \( \rho \), its resource assimilation rate \( a \) (mass per unit area and per unit time), and its resource use rate \( b \) (mass per volume per time). Show that there is a physical law involving two dimensionless quantities only.

1.2 Part II. The heat equation

3. State and prove the maximum principle for the heat equation

\[
\frac{\partial u}{\partial t}(x, y, t) = \Delta u(x, y, t) + f(x, y, t), \quad (x, y) \in \Omega, \ t > 0,
\]

where \( \Omega \subset \mathbb{R}^2 \) is open and bounded, with smooth boundary \( \partial \Omega \).

Provide a physical interpretation of the maximum principle.

4. Show that if

\[
\frac{\partial u}{\partial t} - \kappa \frac{\partial^2 u}{\partial x^2} = 0 \quad \text{for} \quad 0 < x < l,
\]

\[
\frac{\partial u}{\partial x}(0, t) = 0,
\]

\[
\frac{\partial u}{\partial x}(l, t) = 0,
\]

\[
\frac{\partial u}{\partial x}(x, 0) = 0 \quad \text{for} \quad 0 < x < l.
\]

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the maximum of $u$ for $0 < x < l$ and $0 < t < \bar{t}$ must occur at $t = 0$ or at $x = l$.

5. Consider the partial differential equation

\[ \frac{\partial u}{\partial t} = \kappa \Delta u + au \quad \text{for} \quad x \in \Omega \subset \mathbb{R}^3, \]
\[ u(x, 0) = u_0(x) \quad \text{in} \quad \Omega, \]
\[ u = 0 \quad \text{on} \quad \partial \Omega \]

where $\Omega$ is open and bounded.

- Write down the energy law of the problem.
- Find the long-time behavior of the solutions, that is, the limit $u(x, t)$ as $t \to \infty$.

6. Suggested reading assignment from the book by Sam Howison: Chapter 1, Sections 2.1 and 2.2, Sections 3.1-3.3.

The assignment is due on Friday, Sept 23.